NOTE ON ASYMPTOTIC EXPRESSIONS IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS.*

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LET n independent solutions of the linear differential equation

$$(1)\qquad \frac{d^n u}{dx^n} + P_2(x)\frac{d^{n-2}u}{dx^{n-2}} + \cdots + P_n(x)u + \rho^n u = 0$$

be denoted by $y_1, y_2, \cdots, y_n$. It is the aim of this note to establish asymptotic representations of a particular form for the n functions $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$ determined by the n identities

$$(2)\qquad \sum_{i=1}^{n} y_i^{(s-1)}\bar{y}_i = \begin{cases} 0, \text{ if } s = 1, 2, \cdots, n-1, \\ 1, \text{ if } s = n. \end{cases}$$

For this purpose we employ asymptotic forms for the y's, as follows.† If the coefficients $P_s(x)$ in (1) have continuous derivatives of order $(m + n - s)$, m being a positive integer or zero, in the interval $a \leqq x \leqq b$, then there exist n independent solutions of (1) of the form

$$(3)\qquad \begin{aligned} y_i &= u_i(x, \rho) + e^{\rho w_i(x-c)}E_{i0}/\rho^{m+1}, \\ y_i^{(k)} &= u_i^{(k)}(x, \rho) + e^{\rho w_i(x-c)}E_{ik}/\rho^{m+1-k} \\ &\qquad (i = 1, 2, \cdots, n;\ k = 1, 2, \cdots, n-1), \end{aligned}$$

where

$$(4)\qquad u_i(x, \rho) = e^{\rho w_i(x-c)}\left[1 + \frac{\varphi_1(x)}{\rho w_i} + \cdots + \frac{\varphi_m(x)}{(\rho w_i)^m}\right].$$

The functions $\varphi_j(x)$ have continuous $(m + n - j)$th derivatives in (a, b) and are independent of i, while for x in (a, b)

* The formulas given here were published without proof in the *Proceedings Nat. Academy of Sciences*, vol. 2 (1916), pp. 543–5.

† The existence of asymptotic solutions of (1) in nearly the form given in (3) was proved by Birkhoff, *Transactions Amer. Math. Society*, vol. 9 (1908), pp. 219–231, 381–2. The proof of the formulas in (3) and (4) is conducted in a similar manner and offers no essentially new difficulty. For an explicit statement of the difference between Birkhoff's formulas and those here given, see the note in the *Proceedings* referred to above.