CONCERNING ABSOLUTELY CONTINUOUS FUNCTIONS.

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IN a paper "Sulle funzioni integrali" published in 1905 in the *Atti della R. Accademia delle Scienze di Torino,* Vitali defined an important class of functions of limited variation to which he gave the name of *absolutely* continuous functions. He defines these functions as follows:

Let $F(x)$ be a finite function of the real variable x in an interval (a, b), where $a < b$, and let (α, β) be a partial interval of (a, b), $a \leqq \alpha < \beta \leqq b$. Call $F(\beta) - F(\alpha)$ the *increment* of $F(x)$ in (α, β). Call the sum of such increments, if it is finite and determinate, over a group of distinct (α, β)-intervals, the *increment of $F(x)$ in this group;* then, if for every $\sigma > 0$ there exists a $\mu > 0$ such that the modulus of the increment of $F(x)$ over every group of intervals of sum less than μ is less than σ, then $F(x)$ is said to be absolutely continuous in (a, b). Vitali then shows that $F(x)$ is a continuous function of limited variation, while continuous functions of limited variation are *not* all absolutely continuous, and establishes among others the following important

THEOREM. $F(x) - F(a) \equiv \int_a^x \Lambda F(x)dx$, *where* $\int \Lambda F$ *denotes the Lebesgue integral of one of the derivates of $F(x)$; and absolutely continuous functions are the only ones possessing this property.**

Lebesgue had already shown that the derivates of continuous functions of limited variation are summable and that in certain special cases the Lebesgue integral is the primitive function. Vitali's necessary and sufficient condition completes Lebesgue's theory in an important particular and shows that absolutely continuous functions constitute an important generalization of the class of analytic functions, and just as analytic functions can frequently be defined by general descriptive properties it is to be expected that such properties might exist for Vitali's functions. It is the purpose of this paper to show that

* For a proof of this theorem see Vallée Poussin's Cours d'Analyse, Tome 1, § 265, 3d edition.