ON THE SUMMABILITY OF FOURIER'S SERIES.

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1. LET

$$(1)\quad A_n^{(k)} = \frac{(k+1)(k+2)\cdots(k+n)}{n!} = \frac{\Gamma(n+k+1)}{\Gamma(k+1)\Gamma(n+1)} \quad (n=1, 2, 3, \cdots), \quad A_0^{(k)}=1,$$

so that

$$(2)\quad \frac{1}{(1-z)^k} = \sum_{n=0}^{\infty} A_n^{(k-1)} z^n, \quad (|z|<1);$$

then the identity

$$\sum_{n=0}^{\infty} A_n^{(k)} z^n = \frac{1}{(1-z)^{k+1}} = \frac{1}{1-z}\cdot\frac{1}{(1-z)^k} = \sum_{\mu=0}^{\infty} z^\mu \cdot \sum_{\nu=0}^{\infty} A_\nu^{(k-1)} z^\nu$$

gives

$$(3)\quad A_n^{(k)} = \sum_{\nu=0}^{n} A_\nu^{(k-1)} = \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)}.$$

The nth Cesàro mean of order k of a given series $u_0 + u_1 + \cdots + u_n + \cdots$ is, by definition, equal to

$$(4)\quad s_n^{(k)} = \frac{1}{A_n^{(k)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(k)} u_\nu = \frac{1}{A_n^{(k)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} \sum_{\mu=0}^{\nu} u_\mu$$

(both definitions being equivalent on account of (3)), and if $\lim_{n=\infty} s_n^{(k)}$ exists and equals s, the given series is said to be summable by Cesàro's means of order k, or briefly, summable (Ck), with the sum s.

In the present note, I propose to give a simplified proof of the following theorem, due to Riesz and Chapman:*

* M. Riesz, "Sur les séries de Dirichlet et les séries entières," *Comptes rendus de l'Académie des Sciences* (Paris), vol. 149 (1909), pp. 909–912 (gives no details of the proof).

S. Chapman, "Non-integral orders of summability of series and integrals," *Proceedings of the London Mathematical Society*, ser. 2, vol. 9 (1911), pp. 369–409. (See p. 390.)