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NOTE ON THE GAMMA FUNCTION.

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IN this note I wish to present a simple development of the principal properties of the function $\Gamma(x)$, based on the elements of the theory of functions of a complex variable.

1. Let $\varphi(x)$ denote the function

$$x^{x-\frac{1}{2}}e^{-x}\sqrt{2\pi} \quad (-\pi < \arg x < +\pi), \tag{1}$$

where that determination of $\varphi(x)$ is chosen which is real and positive when the complex variable x is real and positive. The function $\Gamma(x)$ is defined to be

$$\lim_{n=\infty} \frac{1}{x} \cdot \frac{1}{x+1} \cdots \frac{1}{x+n} \varphi(x+n+1) \tag{2}$$

$$(n = 0, 1, 2, \cdots ; \; x \neq 0, -1, -2, \cdots).*$$

It is necessary to establish first that the limit of the sequence exists. Denote the $(n+1)$th term of the sequence by $p_n(x)$. It is clear that none of these terms vanish, and that the question of convergence of the sequence is essentially the same as of the series

$$\log p_0(x) + \log [p_1(x)/p_0(x)] + \log [p_2(x)/p_1(x)] + \cdots, \tag{3}$$

where the principal logarithms are taken.

We have at once the relation

$$\frac{p_n(x)}{p_{n-1}(x)} = \frac{\varphi(x+n+1)}{(x+n)\varphi(x+n)}. \tag{4}$$

* A similar formula has been obtained by Enneper, Dissertation, Göttingen (1856), p. 10.