$$(x + y)^p \equiv x^p + y^p \mod p^3,$$

from which (1′) is readily deduced.

Professor Birkhoff points out further that the test fails to be effective for all primes p of the form $6n + 1$. For if $p = 6n + 1$ it follows from the theory of primitive roots modulo p^3 that the congruence

$$t^3 \equiv 1 \mod p^3$$

has a solution t for which $t - 1$ is prime to p. Hence also

$$t^2 + t + 1 \equiv 0 \mod p^3.$$

Then we have

$$(t + 1)^p = (t + 1)(t + 1)^{6n} \equiv (t + 1)(- t^2)^{6n} \equiv t + 1 \mod p^3,$$

$$(t + 1)^{p^2} \equiv (t + 1)^p \equiv t + 1 \mod p^3,$$

and

$$t^p \equiv t \cdot t^{6n} \equiv t \mod p^3, \qquad {}^{p^2} \equiv t^p \equiv t \mod p^3.$$

Therefore

$$(t + 1)^{p^2} \equiv t^{p^2} + 1 \mod p^3.$$

Now put

$$t = \sigma + vp, \qquad (0 < \sigma < p - 1).$$

Then

$$t^{p^2} \equiv \sigma^{p^2}, \quad (t + 1)^{p^2} \equiv (\sigma + 1)^{p^2} \mod p^3.$$

Therefore

$$(\sigma + 1)^{p^2} \equiv \sigma^{p^2} + 1 \mod p^3, \quad (0 < \sigma < p - 1).$$

This is relation (7) of my previous note; from this follows (1) as in the earlier treatment. Hence (1) is satisfied by all primes of the form $6n + 1$. Therefore the test can be useful only when the exponent p is 3 or is of the form $6n - 1$.

INDIANA UNIVERSITY,
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AN EXTENSION OF A THEOREM OF PAINLEVÉ.

BY DR. E. H. TAYLOR.

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THEOREM: Let $f(z)$ be a function which is single-valued and analytic throughout the interior of a region S of the z-plane, $z = x + yi$. If $f(z)$ vanishes at every point of a