TAUTOCHRONES AND BRACHISTOCHRONES.

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IN the simplest case of a particle acted upon by gravity, the tautochrone problem, solved by Huygens in 1673, and the brachistochrone problem, solved by Jean Bernoulli in 1697, give rise to the same curves, namely, cycloids with horizontal bases and concavity upwards. In the case of a general field of force the two problems lead to distinct systems of curves. Their differential equations, each of the third order, are given in the first section of this note. In § 2 it is shown that the only force besides gravity for which the two systems coincide is the central force varying directly as the distance from the origin. If the force generating the brachistochrones is not required to be the same as that generating the tautochrones, then a third case of duplication is possible (§ 4). Incidentally, a class of forces involving eight parameters and related to infinitesimal collineations presents itself (§ 3); they are the only fields of force for which every straight line of the plane is a tautochrone.

§ 1. *General Equations.*

We consider a particle of unit mass moving in the plane under a force whose rectangular components are $\phi(x, y)$, $\psi(x, y)$. With reference to an arbitrary curve the normal and tangential components are

$$(1) \qquad N = \frac{\psi - y'\phi}{\sqrt{1 + y'^2}}, \quad T = \frac{\phi + y'\psi}{\sqrt{1 + y'^2}}.$$

The condition for a tautochrone is that the motion along the curve be harmonic, that is,

$$(2) \qquad T = \kappa(s - s_0),$$

where κ is a constant * and $s - s_0$ denotes the arc reckoned

* For an *actual* tautochrone κ must be negative. The differential equation (3) applies also when κ is positive. Such curves may be termed *virtual* tautochrones. They are the actual tautochrones of the reversed force. A similar remark applies to trajectories, brachistochrones, and catenaries.