ANSWER TO A QUESTION RAISED BY CAYLEY AS REGARDS A PROPERTY OF ABSTRACT GROUPS.

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IN 1859 Cayley * gave an enumeration of the possible abstract groups of order 8, and at the end of the note devoted to this subject he considered briefly the groups defined by two operators s_1, s_2 satisfying the following conditions:

$$s_1^m = 1 \qquad s_2^n = 1 \qquad s_1 s_2 = s_2 s_1^k.$$

He remarks, "the group corresponding to $k = 1$ is distinct from that for any other value of k, but I have not ascertained whether the values other than unity do, or do not, give groups distinct from each other." That different values of k *may* lead to distinct groups is very evident, and we shall assume that the question whose answer Cayley was seeking may be expressed as follows: *Given m and n, and that $k \not\equiv 1 \bmod m$, for what admissible values of k are the groups generated by s_1 and s_2 distinct?* Even if this question should be more general than the one which Cayley had in mind, it relates to such a fundamental matter as to make a direct answer desirable. Partial answers may be found in various places, especially in a comparatively recent paper by Netto, † which is largely devoted to these elementary groups.

The conditions imposed on s_1 and s_2 are equivalent to the conditions that a cyclic group of finite order m is transformed into itself by an operator of finite order n. As a first result we have that the order of G, the group generated by s_1 and s_2, is mn/l, where l is the number of operators common to the two cyclic groups generated by s_1 and s_2 respectively. Cayley implicitly assumed $l = 1$. When $k \equiv 1 \bmod m$, G is either cyclic or the direct product of two cyclic groups. This special case will be excluded in what follows, as it is not included in the question under consideration. As s_1 and s_2 are supposed to be non-commutative, there is some lowest power of s_2, say s_2^r,

* *Philosophical Magazine*, vol. 18 (1859), p. 34.
† *Crelle*, vol. 128 (1905), p. 243.