THE MAXIMUM VALUE OF A DETERMINANT.

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HADAMARD* has shown that the maximum value reached by the modulus of a determinant of order n the moduli of whose elements do not exceed unity is $n^{n/2}$. The result is fundamental in Fredholm's theory of integral equations.†

The problem, when the elements are required to be real, was studied by Professor Hathaway and myself in 1882. I do not think Mr. Hathaway's work was published. My own was published ‡ only in the abstract here reproduced.

"The elements of a determinant being restricted to a variation between the limits $-a$ and $+a$, it is found that for all determinants whose order is greater than 2, a numerical maximum will be obtained by putting the elements $-a$ in the principal diagonal and making all the other elements of the determinant $+a$. If we denote such a determinant of order n when $a=1$ by D_n and the minor of an element in the ith row and kth column by A_{ik}, we have always $A_{ii}=D_{n-1}$, $A_{ik}=-D_{n-1}/(n-3)$, so that

$$D_n = -\left(1+\frac{n-1}{n-3}\right)\cdot D_{n-1} = +\left(1+\frac{n-1}{n-3}\right)\left(1+\frac{n-2}{n-4}\right)D_{n-2}$$

$$= \pm\left(1+\frac{n-1}{n-3}\right)\left(1+\frac{n-2}{n-4}\right)\cdots\left(1+\frac{5}{3}\right)\left(1+\frac{4}{2}\right)\left(1+\frac{3}{1}\right)D_3$$

$$= \pm(n-2)2^{n-1}, \text{ since } D_3 = 4.$$

"The sign will be $\pm$ according as n is $\substack{\text{odd}\\\text{even}}$. The effect of a change in any element is to lessen the greatest factor leaving the rest the same.

"For the maximum cubic determinant $D_n^{(3)}a^n$, we have merely to make all the strata identical and equal to $D_n^{(2)}$. Its value is $\pm n!\,D_n^{(2)}a^n$.

"The four-dimensional determinant may be looked upon as a sort of determinant of plane determinants, the terms of the devel-

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* *Bulletin des sciences math.*, 1893, p. 240. Pascal, I Determinanti, p. 240.

† *Acta Math.*, vol. 27 (1903), p. 365.

‡ Johns Hopkins University Circular, vol. 2, No. 20 (December, 1882), p. 22.