

REMARKS CONCERNING THE VARIATION OF THE LENGTH OF A CURVE.

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(Extract from a letter to Professor Oskar Bolza.)

IN reading through your treatment* of the shortest curve on a given surface it occurred to me that you might like to see the following investigation of the (weak) variation of the length of a curve on a surface :

As usual

$$\delta \int_{t_0}^{t_1} \sqrt{x'^2 + y'^2 + z'^2} dt = \left[\frac{x'}{s'} \delta x + \frac{y'}{s'} \delta y + \frac{z'}{s'} \delta z \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum \left[\delta x \frac{d}{dt} \left(\frac{x'}{s'} \right) \right] dt,$$

and the last integral may be written

$$\int_{s_0}^{s_1} \sum \left(\delta x \frac{d^2 x}{ds^2} \right) ds$$

(from which of course your equations (32) page 130 follow). Now if κ is the curvature of the curve, and if l, m, n are the direction cosines of its principal normal,

$$\frac{d^2 x}{ds^2} = \kappa l, \text{ etc.}$$

Thus the last expression is equal to

$$\int_{s_0}^{s_1} \delta \nu \cdot \kappa \cos \theta \cdot ds,$$

where $\delta \nu$ is the component of $\delta x, \delta y, \delta z$ resolved in a direction † in the tangent plane perpendicular to the tangent to the curve, while θ is the angle between the direction of $\delta \nu$ and the principal normal. Now by the elementary theory of geodesic curva-

* Bolza, Lectures on the Calculus of Variations, pp. 128-130.

† The vector $\delta \nu$ must be drawn to the left of the positive direction of the tangent, as we look down upon the tangent plane from the positive direction of the normal to the surface.