NOTE ON CAUCHY'S INTEGRAL.

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(Read before the American Mathematical Society, December 28, 1903.)

THE analogy between the formula given by Green for a potential function

$$u(x, y) = \frac{1}{2\pi} \int_C u(s) \frac{\partial G}{\partial n} ds \tag{1}$$

and Cauchy's integral representation of a complex function

$$f(z) = \frac{1}{2\pi i} \int_C f(c) \frac{dc}{c - z} dc \tag{2}$$

has been pointed out;* the direct deduction of one from the other may be of interest.

We start with the case where the curve C is a circle of radius 1. Let $z = x + iy = re^{i\vartheta}$ represent the variable point within the circle; let $c = a + ib = \rho e^{is}$ represent a parameter point within or on the circle and $c' = e^{is}/\rho = c/\rho^2$ the reflection of the point c with respect to the circle. Then Green's function for the circle is the real part of $\log [c(z - c')/(z - c)]$, so that if $\Re$ denote "the real part of," the formula (1) may be written

$$u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} u(s) \frac{\partial}{\partial n} \Re \log \frac{c(z - c')}{z - c} ds$$

Noting however that the real and imaginary parts of the logarithm are conjugate functions,† we have, if $v(x, y)$ denote the function conjugate to $u(x, y)$,

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* See the article in the Encl. d. Math. Wiss: "Analytische Functionen complexer Grössen" (p. 17), by Professor Osgood, to whose suggestion this note is due.

† The fact that the derivatives with respect to the normal of a given curve with a determinate tangent of two conjugate functions G and H are still conjugate functions may be verified as follows. G and H satisfy the equations

$$\partial G/\partial x = \partial H/\partial y, \quad \partial G/\partial y = -\partial H/\partial x. \tag{a}$$

If the direction cosines of the given curve be $\cos \alpha(s)$, $\cos \beta(s)$, then by definition

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial a} \cos \beta(s) + \frac{\partial G}{\partial b} \cos \alpha(s),$$