The next meeting of the Section will be held at the University of Chicago on Thursday and Friday, December 28 and 29, 1899.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILL.

AN ELEMENTARY PROOF THAT BESSEL'S FUNCTIONS OF THE ZEROTH ORDER HAVE AN INFINITE NUMBER OF REAL ROOTS.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society at the Meeting of February 25, 1899.)

THE only elementary proof* with which I am acquainted that the function

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots$$

has an infinite number of real roots is the one originally given by Bessel (cf. Gray and Mathews : Treatise on Bessel Functions p. 44). I wish to call attention to a second elementary method of proving this theorem. Although this method is tolerably obvious I do not think it has been used for this purpose before.

In the first place, it is clear from the series for $J_0(x)$ that this function has at least one positive root; for if we substitute in this series first the value $x = 0$, and then the value $x = 3$, we get first a positive and then a negative value. Let us denote the smallest positive root of $J_0(x)$ by c, a quantity whose value can be readily computed as $2.405\cdots$.

We will now prove the theorem :

Any real solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 \tag{1}$$

has an infinite number of real roots.

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* The proofs frequently met with, one depending on the asymptotic value of $J_0(x)$, and the other on what I have called (cf. BULLETIN, vol. 4, p. 298) Sturm's theorem of comparison, cannot be regarded as elementary as they depend on general theorems which can hardly be proved rigorously without some rather delicate analysis.