ON SEVERAL THEOREMS OF OPERATION GROUPS.

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§ 1.

In a recent number of the *Quarterly Journal of Mathematics* (vol. 28, p. 233) we proved the theorem, "Every group (G) whose order is divisible by p^4, p being any prime number, contains a commutative group (G_1) of order p^3." The following proof of this theorem is much simpler and can readily be extended to apply to more general theorems.

G contains a subgroup (G') of order p^α, $\alpha > 3$. G' contains a subgroup of order p whose substitutions* are commutative to all the substitutions of G'.† With respect to this subgroup G' is isomorphic to a group (G_1') of order $p^{\alpha-1}$. G_1' contains a subgroup of order p whose substitutions are commutative to all the substitutions of G_1'. With respect to this subgroup G_1' is isomorphic to a group (G_2') of order $p^{\alpha-2}$. Hence we may suppose the substitutions of G' so arranged that the first $p^\beta (\beta = 0, 1, 2, 3, \cdots, \alpha - 1)$ constitute a self-conjugate (invariant) subgroup of G' and that each of its p sets of $p^{\beta-1}$ substitutions, in order, is transformed into itself by all the substitutions of G'.‡

If we suppose $\beta = 2$ each of the p sets contain p substitutions. The substitutions of $p - 1$ of these sets must be transformed, by all the substitutions of G', according to the cyclical group of order p or according to identity. Those in the first set are known to be transformed according to identity. Hence each of these p^2 substitutions must be commutative to at least $p^{\alpha-1}$ substitutions of G' and the the first p^4 in the given arrangement must contain a commutative group of order p^3. This proves the given theorem.

In general, the first $p^{\beta-1}$ substitutions in the given arrangement are transformed by G' according to a group (H) of order p^θ. A substitution which is commutative to all the substitutions in the second set of $p^{\beta-2}$ is commutative to each of the given $p^{\beta-1}$ substitutions. Hence H is simply isomorphic to a group whose degree cannot exceed $p^{\beta-2}$ and the maximum value (M) of θ is given by the formula §

* The operations are throughout represented by means of substitutions.
† Sylow, *Mathematische Annalen*, vol. 5, p. 588.
‡ Ibid.
§ Cf. Dirichlet-Dedekind, Zahlentheorie, p. 27.