GAUSS'S THIRD PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA.

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It is hoped that the following note may be of interest to some readers of the BULLETIN as indicating the connection between Gauss's third proof that every algebraic equation has a root (*Ges. Werke,* vol. III, p. 59 and p. 107), and those branches of mathematics which have since been developed under the names of the Theory of Functions and the Theory of the Potential. The considerations which follow have doubtless suggested themselves to other readers of Gauss's proof, and it even seems extremely probable that Gauss was led to the discovery of his proof by some method not very different from that here indicated.

Each of the three paragraphs of the present note which are marked with a roman numeral contains a complete proof of the theorem that every algebraic equation has a root. These proofs are arranged in the order of increasing complexity of detail, but of decreasing number of theorems assumed known. The last is essentially Gauss's proof.

Let $f(z) = 0$ be the equation (of the nth degree) for which we wish to prove the existence of a root, and suppose that in the polynomial $f(z)$ the coefficient of z^n is 1. The idea which underlies the proof we shall give is, that if we can prove that $\phi(z)/f(z)$, where $\phi(z)$ is a polynomial, does not remain finite for all values of z, $f(z) = 0$ must have a root. In what follows I let $\phi(z) = zf'(z) = zdf(z)/dz$. We will write $F(z) = \dfrac{zf'(z)}{f(z)} = u(x, y) + iv(x, y)$, where $z = x + yi$. Before taking up the various forms of proof we will note two points: 1st, $u(o, o) = 0$; 2d, if we describe a circle of radius a about the origin, $u(x, y)$ can be made positive at all points on the circumference of this circle by taking a sufficiently large, since $F(\infty) = n$.

Proof I.—$F(z)$ is obviously a monogenic function so that its real part $u(x, y)$ satisfies Laplace's equation. Moreover, if $f(z) = 0$ had no root, u would be finite, continuous, and single valued together with its first derivatives throughout the entire z-plane. We could therefore apply to it the proposition (familiar from the theory of the potential) that the average value of u upon the circumference of any circle is equal to the value of u at the centre. The value of u at the origin, however is zero while the average value of u upon the