APOLAR TRIANGLES ON A CONIC.

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§ 1. *Apolar Triads.*

Take two triangles, or point-triads, in a plane, say T and T', where (attaching complex numbers to the points in the usual way) T is t_1, t_2, t_3, and T' is t_1', t_2', t_3'. Take the polar pair of t_1' as to T, and the polar point of t_2' as to this polar pair, and let this polar point be t_3'. The relation thus imposed on T and T' is symmetric both as to the points T, the points T', and the two triads T and T'; it is, in fact,

$$s_1 s_2' - s_2 s_1' + 3(s_3 - s_3') = 0, \tag{1}$$

where $s_1 = \Sigma t_\lambda$, $s_2 = \Sigma t_\mu t_\nu$, $s_3 = t_1 t_2 t_3$, and similarly for T'. Compare Salmon, Higher Algebra, § 151. Or in the symbolic notation, if T and T' are given by $a_t^3 = 0$, $a'_{t'}{}^3 = 0$, the relation is $(aa')^3 = 0$. The triads are now said to be *apolar.*

This covariant notion of apolarity, stated above for triads, is the natural extension of the notion of harmonic pairs, and can be immediately generalized, as is well known.

If the points lie on a line, we can deal with them projectively. Joining them to any point we have two apolar triads of a pencil. Pass a conic through the vertex of the pencil; the triads of rays cut out apolar point-triads of the conic. Express the co-ordinates of any point of the conic parametrically, say by

$$x : y : z = t^2 : t : 1;$$

then the parameters of the two triads of the conic obey the relation (1), inasmuch as the parameter t is proportional to the ratio in which a side of the triangle of reference is divided by the line joining the opposite vertex of the triangle to the point t of the conic. And in general this is sufficient justification for the interpretation of binary forms and their covariants by means of points of a conic.

Now if in the treatment of covariants by means of complex numbers (which I will call for shortness the inversive method) we restrict our view to points of a circle, any interpretation of a covariant (or rather of its vanishing), so obtained, can be at once stated also projectively for a conic. For if we take the circle as having the equation

$$xz = 1,$$

where x, z are conjugate complex numbers, this equation is a special form of the equation

$$xz = y^2$$