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004

FIXED POINT THEOREMS FROM A DE RHAM PERSPECTIVE*

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1. Introduction. Let M be a smooth compact oriented Riemannian manifold of dimension n, and $f : M \to M$ a smooth map. Define the Lefschetz number

$$L(f) = \sum_{p=0}^{n} (-1)^p \mathrm{Trace}(f^*|_{H^p(M)}).$$

The classical Lefschetz fixed point theorem states that if f has isolated nondegenerate fixed points, then

$$L(f) = \sum_{f(b)=b} \mathrm{sign}\det(\mathrm{df_b} - \mathrm{I}).$$

Atiyah and Bott ([AB1],[AB2]) generalized this theorem to complexes of elliptic operators; we briefly recall (under mild restrictions) their theorem. Let E_0, E_1, $\cdots, E_N$ be a sequence of smooth hermitian vector bundles over M, equipped with a sequence of first order differential operators $D_i : \Gamma(E_i) \to \Gamma(E_{i+1})$. This sequence, denoted $\Gamma(E)$, is called an elliptic complex if for all i,

$$D_{i+1}D_i = 0,$$

and

$$D_i^* D_i + D_{i-1}D_{i-1}^* \text{ is elliptic.}$$

Here we set $D_i = 0$ for $i \notin [0, N-1]$. Set

$$H^p(\Gamma(E)) = Ker D_p / Im D_{p-1}.$$

Given a smooth map f and smooth bundle homomorphisms $\phi_p : (f^*E)_p \to E_p$, we may define endomorphisms $T_p : \Gamma(E_p) \to \Gamma(E_p)$ by

$$T_p s = \phi_p f^* s.$$

When

$$D_p T_p = T_{p+1} D_p, \tag{1.1}$$

$T := (T_0, \cdots, T_n)$ is called a geometric endomorphism of the complex $\Gamma(E)$. It induces endomorphisms H^pT of $H^p(\Gamma(E))$, and we define the Lefschetz number

$$L(T) = \sum_{p=0}^{N} (-1)^p \mathrm{Trace}\, H^pT.$$

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