ASIAN J. MATH.
Vol. 11, No. 3, pp. 497–534, September 2007

PATTERNS GENERATION AND SPATIAL ENTROPY IN TWO-DIMENSIONAL LATTICE MODELS*

JUNG-CHAO BAN†, SONG-SUN LIN‡, AND YIN-HENG LIN§

Abstract. Patterns generation problems in two-dimensional lattice models are studied. Let $\mathcal{S}$ be the set of p symbols and $\mathbb{Z}_{2\ell\times 2\ell}$, $\ell \geq 1$, be a fixed finite square sublattice of $\mathbb{Z}^2$. Function $U : \mathbb{Z}_{2\ell\times 2\ell} \to \mathcal{S}$ is called local pattern. Given a basic set $\mathcal{B}$ of local patterns, a unique transition matrix $\mathbb{A}_2$ which is a $q^2 \times q^2$ matrix, $q = p^{\ell^2}$, can be defined. The recursive formulae of higher transition matrix $\mathbb{A}_n$ on $\mathbb{Z}_{2\ell\times n\ell}$ have already been derived [4]. Now $\mathbb{A}_n^m$, $m \geq 1$, contains all admissible patterns on $\mathbb{Z}_{(m+1)\ell\times n\ell}$ which can be generated by $\mathcal{B}$. In this paper, the connecting operator $\mathbb{C}_m$, which comprises all admissible patterns on $\mathbb{Z}_{(m+1)\ell\times 2\ell}$, is carefully arranged. $\mathbb{C}_m$ can be used to extend $\mathbb{A}_n^m$ to $\mathbb{A}_{n+1}^m$ recursively for $n \geq 2$. Furthermore, the lower bound of spatial entropy $h(\mathbb{A}_2)$ can be derived through the diagonal part of $\mathbb{C}_m$. This yields a powerful method for verifying the positivity of spatial entropy which is important in examining the complexity of the set of admissible global patterns. The trace operator $\mathbb{T}_m$ of $\mathbb{C}_m$ can also be introduced. In the case of symmetric $\mathbb{A}_2$, $\mathbb{T}_{2m}$ gives a good estimate of the upper bound on spatial entropy. Combining $\mathbb{C}_m$ with $\mathbb{T}_m$ helps to understand the patterns generation problems more systematically.

Key words. Lattice dynamical systems, Spatial entropy, Patterns generation, Connecting operator, Trace operator

AMS subject classifications. Primary 37B50; Secondary 37B40

1. Introduction. Lattices are important in scientifically modelling underlying spatial structures. Investigations in this field have covered phase transition [11], [12], [34], [35], [36], [37], [38], [45], [46], [47], [48], chemical reaction [7], [8], [24], biology [9], [10], [21], [22], [23], [31], [32], [33] and image processing and pattern recognition [16], [17], [18], [19], [20], [25]. In the field of lattice dynamical systems (LDS) and cellular neural networks (CNN), the complexity of the set of all global patterns recently attracted substantial interest. In particular, its spatial entropy has received considerable attention [1],[2], [3], [4], [5], [13], [14], [15], [28], [29],[30], [39], [40], [41], [42], [43], [44].

The one dimensional spatial entropy h can be found from an associated transition matrix $\mathbb{T}$. The spatial entropy h equals $\log \rho(\mathbb{T})$, where $\rho(\mathbb{T})$ is the maximum eigenvalue of $\mathbb{T}$.

In two-dimensional situations, higher transition matrices have been discovered in [30] and developed systematically [4] by studying the patterns generation problem.

This study extends our previous work [4]. For simplicity, two symbols on 2×2 lattice $\mathbb{Z}_{2\times 2}$ are considered. A transition matrix in the horizontal (or vertical)

*Received October 18, 2005; accepted for publication July 25, 2006. This reserch is partially supported by the National Science Council, R.O.C.(Contract No. NSC 94-2115-M-009-002) and the National Center for Theoretical Sciences.

†The National Center for Theoretical Sciences, Hsin-Chu 300, Taiwan (jcban@math.cts.nthu.edu.tw).

‡Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan (sslin@math.nctu.edu.tw).

§Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan (yhlin.am91g@nctu.edu.tw).