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THE OVERCONVERGENT FROBENIUS*

ROBERT F. COLEMAN†

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We will improve some estimates of Dwork and Gouvêa concerning the the U-operators on overconvergent forms of integral weight. One consequence of our estimates that is not evident from earlier results is that the U-operator applied to an overconvergent form of integral weight bounded by one on a neighborhood of the ordinary locus is still bounded by one on a neighborhood of the ordinary locus.

Let K be a complete local field contained in $\mathbf{C}_p$ with ring of integers R_K. Fix N, $(N,p)=1$. For $r \in R_K$, $Z(N,r)$ will denote the affinoid subdomain of $X_1(N)$ defined over K where $|E_{p-1}| \geq |r|$ (so a neighborhood of the component of the ordinary locus containing the cusp ∞). Let ϕ: $Z(N,r) \to Z(N,r^p)$ be the canonical Frobenius, which is defined when $v(r) < 1/(p+1)$. Let $S(N,r) := S(R_K,N,r)$ denote the R_K-module of forms of weight 0 on $Z(N,r)$ of absolute value at most 1, $S(K,N,r) = S(R_K,N,r) \otimes K$, $Z(r) = Z(1,r)$ and $S(r) = S(1,r)$. For $\alpha \in R_K/pR_K$, we set $v(\alpha) = v(\tilde{\alpha})$ for any $\tilde{\alpha} \in R_K$ which reduces to α, if $\alpha \neq 0$ and $v(\alpha) = \infty$ otherwise.

PROPOSITION 1. *When $N = 1$, ϕ is defined on $Z(r)$, $v(r) < p/(p+1)$. Let $h(j)$ denote the Hasse invariant of any elliptic curve modulo p with j-invariant j mod p. Then*

(i) $$|\phi(j) - j^p| \leq |p/h(j)|$$

(ii) $$Tr_\phi(S(r)) \subseteq pr^{-(p+1)}S(r^p).$$

Proof. For a supersingular point e let $i_e = 3$ if $j(e) = 0$, $i_e = 2$ if $j(e) = 1728$ and $i_e = 1$ otherwise. Dwork asserts, at formula (7.8) of "p-adic Cycles," that

$$\phi(j) = j^p + pk(j) + \sum_e \sum_{n=1}^{\infty} \frac{A_{e,n}}{(j-\beta_e)^n}$$

where $k(j)$ is a polynomial in j of degree at most $p-1$ over $\mathbf{Z}_p$, e runs over the supersingular points modulo p, β_e is a point in the residue class above e defined over $\mathbf{Q}_p^{unr}$ such that $\beta_{\bar{a}} = a$ when $a = 0$ or 1728 and $A_{e,n} \in \mathbf{Q}_p^{unr}$ such that

$$v(A_{e,n}) \geq \frac{1}{p+1} + i_e n \left(\frac{p}{p+1}\right).$$

Now $v(j - \beta_e) = i_e v(h(j))$, if $e = \bar{j}$ is supersingular and $0 < v(h(j)) < 1$.

Thus

$$v\left(\frac{A_{e,n}}{(j-\beta_e)^n}\right) \geq 1 + (ni_e - 1)(\frac{p}{p+1} - v(h(j))) - v(h(j)),$$

*Received February 25, 2012; accepted for publication June 26, 2012.
†Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA (coleman@math.berkeley.edu).