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009

FLOER COHOMOLOGY AND DISC INSTANTONS OF LAGRANGIAN TORUS FIBERS IN FANO TORIC MANIFOLDS*

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Abstract. In this paper, we first provide an explicit description of *all* holomorphic discs ("disc instantons") attached to Lagrangian torus fibers of arbitrary compact toric manifolds, and prove their Fredholm regularity. Using this, we compute Fukaya-Oh-Ohta-Ono's (FOOO's) obstruction (co)chains and the Floer cohomology of Lagrangian torus fibers of Fano toric manifolds. In particular specializing to the formal parameter $T^{2\pi} = e^{-1}$, our computation verifies the folklore that FOOO's obstruction (co)chains correspond to the Landau-Ginzburg superpotentials under the mirror symmetry correspondence, and also proves the prediction made by K. Hori about the Floer cohomology of Lagrangian torus fibers of Fano toric manifolds. The latter states that the Floer cohomology (for the parameter value $T^{2\pi} = e^{-1}$) of all the fibers vanish except at a finite number, the Euler characteristic of the toric manifold, of base points in the momentum polytope that are critical points of the superpotential of the Landau-Ginzburg mirror to the toric manifold. In the latter cases, we also prove that the Floer cohomology of the corresponding fiber is isomorphic to its singular cohomology.

We also introduce a restricted version of the Floer cohomology of Lagrangian submanifolds, which is a priori more flexible to define in general, and which we call the *adapted Floer cohomology*. We then prove that the adapted Floer cohomology of any non-singular torus fiber of Fano toric manifolds is well-defined, invariant under the Hamiltonian isotopy, which is isomorphic to the Bott-Morse Floer cohomology of the fiber.

Key words. Floer cohomology, Lagrangian submanifold, holomorphic disc, toric manifold, Landau-Ginzburg model.

AMS subject classifications. 53D12, 53D40, 14J45, 14J32.

1. Introduction. Floer cohomology of Lagrangian intersections was introduced by Floer [Fl] in symplectic geometry. Since then, its construction has been further generalized [O1] and an obstruction theory to its definition has been developed by Fukaya-Oh-Ohta-Ono [FOOO]. It has been proven to be a powerful tool in studying various problems in symplectic geometry (see [Fl], [O4], [Che], [P], [Se], [FOOO], [BC], and [TY], for example). The theory itself was greatly enhanced by the advent of the Fukaya category [Fu1] and the homological mirror symmetry proposal by Kontsevich [Ko], and also by the open string theory of D-branes in many physics papers, among which [HV], [H] will be the most relevant to the content of the present paper.

Even in the midst of these theoretical enhancement and successful applications of the Floer theory, actual computation of Floer cohomology itself for specific examples remains to be a non-trivial task, especially with $\mathbb{Z}$-coefficients (not just with $\mathbb{Z}_2$-coefficients), except for the cases where there is no quantum contribution [Fl] or for the case of *real manifolds* i.e., the fixed point sets of anti-holomorphic involutions [O2], [FOOO]. Indeed, computation of the Floer cohomology in the presence of nontrivial holomorphic discs requires detailed understanding of the quantum contribution of the holomorphic discs (or the effect of "open string instantons" in the physics terminology) to the cohomology of the Lagrangian submanifolds. In this respect, the recent

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