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CANONICAL METRICS ON 3-MANIFOLDS AND 4-MANIFOLDS*

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In Memory of S. S. Chern

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1. Introduction. In this paper, we discuss recent progress on the existence of canonical metrics on manifolds in dimensions 3 and 4, and the structure of moduli spaces of such metrics. The existence of a "best possible" metric on a given closed manifold is a classical question in Riemannian geometry, attributed variously to H. Hopf and R. Thom, see [22] for an interesting perspective. A good deal of motivation for this question comes from the case of surfaces; the uniformization theorem in dimension 2 has a multitude of consequences in mathematics and physics. Further, there are strong reasons showing that the closest relations between geometry and topology occur in dimensions 2, 3 and 4.

The precise formulation of the question in dimension 3 is given by Thurston's Geometrization Conjecture. This conjecture describes completely when a given 3-manifold admits a canonical metric (defined to be a metric of constant curvature or more generally a locally homogeneous metric), and thus determines exactly what the obstructions are to the existence of such a metric. Moreover, it describes how an arbitrary 3-manifold decomposes into topologically essential pieces, each of which admits a canonical metric, resulting in the topological classification of 3-manifolds. The apparent solution of the Geometrization Conjecture by Perelman is one of the most spectacular breakthroughs in geometry and topology in the past several decades.

The Thurston picture will be reviewed in more detail in §2, for the light it sheds on what might be hoped for or expected in dimension 4. Since there is already considerable analysis and discussion of the details of Perelman's work elsewhere, we will not discuss this in any detail here. We do however give one application of his work, (since this does not seem to be widely known), namely the determination of the value of the Yamabe invariant or Sigma constant $\sigma(M)$ of all 3-manifolds M for which $\sigma(M) \leq 0$, cf. §2.

Thus, the bulk of the paper concerns dimension 4. Canonical metrics will be defined to be metrics minimizing, (or possibly just critical points for), one of the classical and natural curvature functionals $\mathcal{F}$ on the space of metrics $\mathbb{M}$ on a given oriented 4-manifold M:

$$\mathcal{R}^2,\ \mathcal{W}^2,\ \mathcal{W}^2_+,\ \mathcal{W}^2_-,\ \mathcal{R}ic^2. \tag{1.1}$$

These are respectively the square of L^2 norm of the Riemann curvature R, Weyl curvature W, its self-dual and anti-self-dual components, W_+, W_-, and Ricci curvature Ric. We will also consider, but in much less detail, the scalar curvature functionals

$$\mathcal{S}^2, -\mathcal{S}|_{\mathcal{Y}}, \tag{1.2}$$

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