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EIGENFUNCTIONS OF THE LAPLACIAN ON COMPACT RIEMANNIAN MANIFOLDS*

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0. Introduction. Suppose that M is a compact Riemannian manifold. The Laplace operator Δ is given in local coordinates by $\Delta f = g^{-1/2}\frac{\partial}{\partial x_i}\left(g^{1/2}g^{ij}\frac{\partial f}{\partial x_j}\right)$, where g_{ij} denotes the metric tensor and $g = \det(g_{ij})$. It follows from the compactness of M and the theory of elliptic partial differential equations that Δ has pure point spectrum. This means that L^2M admits an orthonormal basis consisting of eigenfunctions ϕ_i of Δ with associated eigenvalues λ_i, that is $\Delta\phi_i = -\lambda_i\phi_i$. There have been many works concerning the eigenvalues and their relationship to the geometry of the manifold. These studies pertain to upper and lower bounds for eigenvalues and asymptotics of eigenvalues. The research concerning the eigenfunctions themselves is less highly developed.

The present paper constitutes an exposition of some topics of interest in current research about eigenfunctions. One source of inspiration is the mathematical physics of quantum chaos, which is concerned with the quantization of classically ergodic systems. The first two sections of the paper discuss concentration properties of eigenfunctions. Concentration is measured both through bounds of the supremum norm and via weak limits of the associated densities $|\phi_i|^2 dvol$, as $i \to \infty$. Theorem 2.3 is a fundamental result concerning the weak limits of these densities on manifolds with ergodic geodesic flow. This theorem of quantum ergodicity has been studied extensively by several mathematicians. Our third section concerns the nodal sets of eigenfunctions. The main focus is upon the conjecture of Yau concerning the Hausdorff measure of the nodal sets. We include some motivating ideas which may be absent from the more technical presentations appearing in the published record.

This article was solicited by the editors for the memorial volume dedicated to S. S. Chern. It may therefore be appropriate to mention that the author received his doctorate in 1974 under the direction of Professor Chern. The thesis consisted of various results about Chern Simons invariants. However, Professor Chern encouraged his student to learn about other topics of mathematical research. These topics included the study of eigenvalues and eigenfunctions of the Laplace operator for Riemannian manifolds.

1. Bounds for the Supremum Norm of Eigenfunctions. Let M be a compact Riemannian manifold and Δ its Laplacian acting on functions. Suppose that ϕ is an eigenfunction of $-\Delta$ with eigenvalue $\lambda \neq 0$, $\Delta\phi = -\lambda\phi$. If one scales the metric by $g_{ij} \to \lambda g_{ij}$, an elliptic equation with bounded coefficients is obtained. Also, a geodesic ball of radius $c\lambda^{-1/2}$ scales to a ball of radius c. Elementary local elliptic theory shows that the L^∞ norm of ϕ is bounded by its L^2 norm relative to the scaled

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