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A CAVEAT ON THE CONVERGENCE OF THE RICCI FLOW FOR PINCHED NEGATIVELY CURVED MANIFOLDS*

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In his seminal paper [5], Hamilton initiated the Ricci flow method for finding Einstein metrics on a closed smooth n-dimensional manifold M^n starting with an arbitrary smooth Riemannian metric h on M^n. He considered the evolution equation

$$\frac{\partial}{\partial t}\, h = \frac{2}{n}\, r\, h \, - \, Ric$$

where $r = \int R\, d\mu / \int d\mu$ is the average scalar curvature (R is the scalar curvature) and Ric is the Ricci curvature tensor of h. Hamilton then spectacularly illustrated the success of this method by proving, when $n = 3$, that if the initial Riemannian metric has strictly positive Ricci curvature it evolves through time to a positively curved Einstein metric h_∞ on M^3. And, because $n = 3$, such a Riemannian metric automatically has constant sectional curvature; hence (M^3, h_∞) is a spherical space-form; i.e. its universal cover is the round sphere. Following Hamilton's approach G. Huisken [6], C. Margerin [7] and S. Nishikawa [9] proved that, for every n, sufficiently pinched to 1 n-manifolds (the pinching constant depending only on the dimension) can be deformed, through the Ricci flow, to a spherical-space form.

Ten years later R. Ye [10] studied the Ricci flow when the initial Riemannian metric h is negatively curved and proved that a negatively curved Einstein metric is strongly stable; that is, the Ricci flow starting near such a Riemannian metric h converges (in the C^∞ topology) to a Riemannian metric isometric to h, up to scaling. (We introduce the notation $h \equiv h'$ for two Riemannian metrics that are isometric up to scaling.) In [10] R. Ye also proved that sufficiently pinched to -1 manifolds can be deformed, through the Ricci flow, to hyperbolic manifolds, but the pinching constant in his theorem depends on other quantities (e.g the diameter or the volume). Ye's paper was motivated by the problem on whether the Ricci flow can be used to deform every sufficiently pinched to -1 Riemannian metric to an Einstein metric (the pinching constant depending only on the dimension). We would also like to mention the paper of Min-Oo about deforming almost Einstein metrics of negative scalar curvature to Einstein metrics [8].

In this short note we show that our previous results [4] imply the existence of pinched negatively curved metrics for which the Ricci flow does not converge in the C^2 topology (hence in the C^k topology, $2 \leq k \leq \infty$) to a negatively curved Einstein

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