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# TORUS BUNDLES AND THE GROUP COHOMOLOGY OF $GL(N, \mathbb{Z})$

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**Abstract**

We prove the vanishing of a certain characteristic class of flat vector bundles when the structure groups of the bundles are contained in $GL(N, \mathbb{Z})$. We do so by explicitly writing the characteristic class as an exact form on the base of the bundle.

In this paper we consider certain characteristic classes of flat complex vector bundles, which are known in algebraic K-theory as the Borel regulator classes. We prove that if the structure group of a rank-$N$ vector bundle is contained in $GL(N, \mathbb{Z})$ with $N$ odd, then the Borel class of degree $2N-1$ vanishes. Our proof is analytic in nature and is a special, but interesting, case of a more general theorem concerning the direct images of flat vector bundles under smooth submersions [4].

The background to our result is the following. First, let $N$ be a positive even integer and let $E$ be a real oriented rank-$N$ vector bundle over a connected manifold $B$. Then the rational Euler class $\chi_{\mathbb{Q}}(E)$ is an element of $\mathrm{H}^N(B; \mathbb{Q})$. Sullivan showed that if $E$ is a flat vector bundle whose structure group is contained in $SL(N, \mathbb{Z})$, then $\chi_{\mathbb{Q}}(E) = 0$ [12]. Let $\Lambda$ be the integer lattice in $E$. Then $M = E/\Lambda$ is the total space of a torus bundle over $B$. Sullivan's proof was by a simple topological argument involving this torus bundle.

Bismut and Cheeger observed that Sullivan's result follows from the Atiyah-Singer families index theorem, applied to the vertical signature operators on the torus bundle [2]. They also showed that one can write

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