J. DIFFERENTIAL GEOMETRY
40 (1994) 311-377

ETA INVARIANTS AND MANIFOLDS WITH BOUNDARY

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0. Introduction

Let M be a compact oriented Riemannian manifold of dimension n, and let S be a Hermitian vector bundle over M. Let $D\colon C^\infty(M, S) \to C^\infty(M, S)$ be a first-order elliptic differential operator on M which is formally selfadjoint with respect to the natural inner product defined by the fibre metric of S and the metric of M. For the moment suppose that M has no boundary. Then D is essentially selfadjoint in $L^2(M, S)$, and the eta invariant is a nonlocal spectral invariant of D, which was introduced by Atiyah, Patodi, and Singer [1]. Let us recall the definition of the invariant. Let λ_j run over the eigenvalues of D. Then the eta function of D is defined as

$$\eta(s, D) = \sum_{\lambda_j \neq 0} \frac{\operatorname{sign}\lambda_j}{|\lambda_j|^s}, \qquad \operatorname{Re}(s) > n. \tag{0.1}$$

The series is absolutely convergent in the half-plane $\operatorname{Re}(s) > n$ and admits a meromorphic continuation to the whole complex plane. The analytic continuation is based on the following alternative expression for the eta function

$$\eta(s, D) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty t^{(s-1)/2} \operatorname{Tr}(De^{-tD^2})\, dt. \tag{0.2}$$

It is a nontrivial result that $\eta(s, D)$ is regular at $s = 0$ [3], [13]. Then the eta invariant is defined to be $\eta(0, D)$. The eta invariant is a measure of the spectral asymmetry of D. It arises naturally as the boundary correction term in the index theorem for manifolds with boundary proved by Atiyah, Patodi, and Singer [1]. We note that this index theorem can be recovered in many different ways. For example, one may glue a half-cylinder or a cone to the boundary of the manifold in question and work in the L^2-setting [7], [22], [23]. This means that the spectral boundary conditions

Received June 18, 1993.