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HOMOGENEOUS KOSZUL MANIFOLDS IN $\mathbf{C}^n$

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0. Introduction

Let X be a complex manifold. We shall say that X is homogeneous under the real analytic Lie group G if X is a homogeneous G-space for which the mapping $\nu: G \times X \to X$ is real analytic in the G variable and holomorphic in the X variable. Suppose that there is a G-invariant volume form ω on X. In local coordinates, we may express ω as

$$\sqrt{-1}/2K(z, \overline{z})\, z_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n .$$

Homogeneity implies that K is strictly positive. In [6], Koszul introduced the following Hermitian form which we refer to as the Koszul form:

$$H = \sum \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log K \, dz_i \, d\overline{z}_j . \tag{1}$$

The form H is invariant under any biholomorphic mapping which preserves the volume form. We shall say that X is a Koszul manifold if H is nondegenerate. This gives X the structure of a pseudo-Kähler manifold for which the measure preserving biholomorphisms are isometries.

In [6], Koszul proposed the problem of the classifying all Koszul manifolds. Considerable progress has been made on this problem in the cases when the Koszul structure is in fact Kähler (see [3] and the references contained therein) and in the symmetric case. In the general pseudo-Kähler case, it seems that very little progress has been made. In this work, we begin the study of this problem. We restrict to the case that G contains an exponential solvable group which acts transitively on X. We refer to such manifolds as "type E." (Note that all Hermitian symmetric spaces are type E. Also, all rationally homogeneous, contractible domains are type E.)

Our first main result includes the statement that all type E manifolds have realizations as homogeneous domains in $\mathbf{C}^n$. The result, however,

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