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ISOSPECTRALITY IN THE FIO CATEGORY

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0. Introduction

Compact riemannian manifolds (M_1, g_1), resp. (M_2, g_2), are called isospectral if there exists a unitary operator $U\colon L^2(M_1) \to L^2(M_2)$ which intertwines their Laplacians: $U\Delta^{(1)}U^* = \Delta^{(2)}$. At this time, quite a variety of (nonisometric) isospectral pairs have been constructed. On the other hand, all of these pairs are quite special: to the author's knowledge, each known pair has a common riemannian cover, and frequently a common quotient. These observations raise the questions:

(Q1)—Are isospectral manifolds locally isometric? Do they have a common riemannian cover?[1]

(Q2)—Is a generic metric spectrally determined (i.e., not nontrivially isospectral to another)? Is a metric with simple length spectrum spectrally determined?

There exist few positive results on these problems at present. Our purpose in this paper is to show that they can be solved (affirmatively) if we restrict the isospectral problem to the FIO (Fourier Integral Operator) category. At least, we will show this for (M, g) of dimension $d = 2$ and curvature $K < 0$. These dimension and curvature restrictions represent the current state of knowledge on the isometry problem for conjugate geodesic flows ([3], [4], [17]; see below); they should become relaxed as this knowledge develops further.

Isospectral Laplacians Δ_1 and Δ_2 will be called isospectral in the FIO category (or, Fourier-isospectral for short) if there exists a unitary FIO U intertwining them as above. More precisely, U will be assumed to lie in the Hörmander space $I^0(M_1 \times M_2, C)$ for some closed, embedded canonical relation $C \hookrightarrow \dot{T}^*M_1 \times \dot{T}^*M_2^-$, such that $C \circ C^t$ is a clean composition (see §1). To prevent confusion, we emphasize that C is not assumed to be the graph of a symplectic diffeomorphism (even locally). Indeed, our first step (§2–3) will be to characterize the canonical relations

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[1] A counterexample has recently been found by C. Gordon.