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LOCAL DIFFERENTIAL GEOMETRY AND GENERIC PROJECTIONS OF THREEFOLDS

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The purpose of this note is to prove a result concerning the 4-secant lines of a nondegenerate irreducible, say smooth, threefold

$$X \subset \mathbf{P}^r, \qquad r \geq 9;$$

namely we prove essentially that all these lines together fill up at most a fourfold (see Theorem 1 below); equivalently, the generic projection of X to $\mathbf{P}^4$ has no fourfold points that come from collinear quadruples of points on X.

The (very classical) subject of generic projections of n folds to $\mathbf{P}^{n+1}$ and the multiple points of such projections has recently come into focus in connection with work of Pinkham [4], Lazarsfeld [2], and Peskine [3], which has shown how certain properties (both known and conjectural) of such projections can be used to establish various cohomological properties of the n folds in question, in particular Castelnuovo regularity. Indeed, Lazarsfeld's paper [2] shows, among other things, that the above statement concerning fourfold points of projections to $\mathbf{P}^4$ is exactly what is needed to establish a sharp Castelnuovo regularity bound for smooth nondegenerate threefolds in $\mathbf{P}^r$, $r \geq 9$ (see Corollary 3 below).

We now proceed with a precise statement.

Theorem 1. *Let X be an irreducible nondegenerate three-dimensional subvariety of $\mathbf{P}^r$, $r \geq 9$, whose tangent variety is six-dimensional, and let $\{L_y : y \in Y\}$ be a family of lines in $\mathbf{P}^r$ with the property that for any general $y \in Y$, the part of the scheme-theoretic intersection $L_y \cap X$ supported at smooth points of X has length at least 4. Then we have*

$$\dim\left(\bigcup_{y \in Y} L_y\right) \leq 4.$$

Remarks 2.1. Any *smooth* threefold has six-dimensional tangent variety (cf. [1]). The hypothesis that X has six-dimensional tangent variety

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