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STABILITY OF HARMONIC MAPS OF KÄHLER MANIFOLDS

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1. Introduction

Harmonic maps [6] φ between Riemannian manifolds (M, g) and (N, h) are the critical points of the Dirichlet energy integral

$$E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 \, d\,\mathrm{vol}.$$

When M and N are Hermitian manifolds Lichnerowicz [9] observed that the energy E decomposes into two parts E' and E'' corresponding with the parts of the differential $d\varphi$ acting in holomorphic and antiholomorphic tangents. The difference $E' - E''$ can be expressed in terms of the Kähler forms ω^M and ω^N as

$$E'(\varphi) - E''(\varphi) = \int_M \langle \varphi^* \omega^N, \omega^M \rangle \, d\,\mathrm{vol},$$

and this is a homotopy invariant provided that ω^M is coclosed and ω^N is closed. In this case maps for which either E'' or E' vanishes are absolute minima of the energy in their homotopy class and hence are stable harmonic maps. They are of course the holomorphic and antiholomorphic maps between M and N. For simplicity we shall refer to them as $\pm$holomorphic maps. These remarks apply in particular to the case where M and N are both Kähler manifolds.

In general we shall say that a harmonic map φ is (weakly) stable if the second variation of the energy is nonnegative:

$$H_\varphi(v, v) = \frac{d^2}{dt^2} E(\varphi_t)|_{t=0} \geq 0$$

for all smooth variations φ_t of φ, where $v = \dot{\varphi}_0$. It may be conjectured that there are no other stable harmonic maps between Kähler manifolds besides the $\pm$holomorphic ones. The problem has recently received a lot

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