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NODAL SETS FOR SOLUTIONS OF ELLIPTIC EQUATIONS

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Here we study, on a connected domain $\Omega \subset \mathbf{R}^n$, the zero set $u^{-1}\{0\}$ of a solution u of an elliptic equation

$$a_{ij}D_iD_ju + b_jD_ju + cu = 0,$$

where a_{ij}, b_j, c are bounded and a_{ij} is continuous.

Our principal result (precisely stated in Theorem (1.7) below) is that the $(n-1)$-dimensional Hausdorff measure of $u^{-1}\{0\}$ is finite in a neighborhood of any point $x_0 \in \Omega$ at which u has finite order of vanishing. (For Lipschitz a_{ij} this holds at *each* point $x_0 \in \Omega$ by the unique continuation theory for elliptic equations.) We actually obtain an explicit bound on the Hausdorff measure of $u^{-1}\{0\}$ in terms of the order of vanishing of u, the modulus of continuity of a_{ij}, and the bounds on a_{ij}, b_j, c.

Notice that in the case the coefficients a_{ij}, b_j, c are analytic, u is then real analytic [8], and the finiteness of the $(n-1)$-dimensional Hausdorff measure of $u^{-1}\{0\}$ is automatic [3, 3.4.8]. The explicit bound on the $(n-1)$-dimensional Hausdorff measure is nevertheless of interest in this case, but a more precise estimate for the real analytic case was already established in [2].

We also show here (in Theorem (1.10)) that if the coefficients are sufficiently smooth then $u^{-1}\{0\}$ decomposes into a disjoint union of the embedded C^1 submanifold $u^{-1}\{0\} \cap \{|Du| > 0\}$ together with the closed set $u^{-1}\{0\} \cap |Du|^{-1}\{0\}$, which we show is countably $(n-2)$-rectifiable. L. Caffarelli and A. Friedman showed already in [1] that $\dim u^{-1}\{0\} \cap |Du|^{-1}\{0\} \leq n-2$ in the case of equations of the special form $\Delta u + f(x,u) = 0$. We thank F. H. Lin for pointing out this reference.

In §5 of the present paper we apply the main estimates of §1 and an estimate of Donnelly and Fefferman [2] for the order of vanishing of eigenfunctions to give an asymptotic bound of the $(n-1)$-dimensional measure

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