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ON THE EXISTENCE AND REGULARITY OF FUNDAMENTAL DOMAINS WITH LEAST BOUNDARY AREA

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Introduction

Let M be a three-dimensional compact smooth Riemannian manifold. Let Φ_0 be the set of all fundamental domains of M with Lipschitz boundary in $\tilde{M}$, the universal covering space of M. Then it is a question of basic interest to see whether one can find a fundamental domain in Φ_0 with least boundary area among all fundamental domains in Φ_0. Moreover, passing to subfamilies of Φ_0, one can ask similar questions: Let Φ_1 be the subfamily of Φ_0 consisting of all fundamental domains of M which are homeomorphic to an open ball, and let Φ_2 be the subfamily of Φ_1 consisting of all fundamental domains of M whose closures are homeomorphic to a closed ball. Can one find a fundamental domain in Φ_1, or Φ_2, whose boundary area (counting multiplicity) is the minimum among all fundamental domains in Φ_1, or Φ_2? These problems were proposed by Michael H. Freedman.

In this paper we answer the first problem, the case of Φ_0, in the affirmative (Theorem 3). We then discuss the second problem, the case of Φ_1, and derive an affirmative answer under the assumption that M is irreducible, that is, every embedded sphere in M bounds a ball (Theorem 5). The third problem, the case of Φ_2, remains open. Besides the existence of minimizing fundamental domains in Φ_0 and Φ_1, we also obtain the regularity of the boundaries of these minimizing fundamental domains (Theorem 4). If we define a spine to be a subset of M whose complement in M is homeomorphic to an open ball, then the second problem is equivalent to finding an area minimizing spine of M.

For a two-dimensional compact Riemannian manifold M^2 the problem is much simpler to solve and easier to visualize. In fact, any fundamental domain of M^2 with least boundary length among all fundamental domains is always homeomorphic to an open disk. Furthermore the boundary of a minimizing fundamental domain consists of geodesic segments of $\tilde{M}^2$ meeting each other at 120° angles, and the number of edges and vertices are both $6 - 6\chi(M)$

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