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THE LORENTZIAN SPLITTING THEOREM WITHOUT THE COMPLETENESS ASSUMPTION

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1. Introduction

A number of papers ([6], [3], [4], [2]) have been published which address the problem posed by Yau [10] of establishing a Lorentzian analogue of the Cheeger-Gromoll splitting theorem of Riemannian geometry. A very satisfactory Lorentzian analogue has recently been obtained by Eschenburg. In [4], he proves that a globally hyperbolic, timelike geodesically complete space-time satisfying the "strong energy condition", $\mathrm{Ric}(X,X) \geq 0$, X timelike, which contains a (complete) timelike line, "splits" in a sense made precise below. Prior to Eschenburg's work, Beem et al. [3] proved a Lorentzian splitting theorem assuming a more stringent sectional curvature condition (analogous to nonnegative sectional curvature in the Riemannian case). One interesting feature of their result is that the full assumption of timelike geodesic completeness is not needed; it is only required that the given timelike line be complete. Timelike geodesic completeness is then derived as a consequence of the assumption of global hyperbolicity, the sectional curvature condition, and the completeness of the line. This suggests that there may be some redundancy in the hypotheses of Eschenburg's theorem.

The purpose of this paper is to prove the Lorentzian splitting theorem for globally hyperbolic space-times obeying the strong energy condition, without the assumption of timelike geodesic completeness; i.e. our aim is to prove the following

Theorem. *Let (M,g) be a connected globally hyperbolic space-time which satisfies $\mathrm{Ric}(X,X) \geq 0$ for all timelike vectors X. If (M,g) contains a complete timelike line γ then it is isometric to $(\mathbb{R}\times S, -dt^2 \oplus h)$, where (S,h) is a complete Riemannian manifold, and the factor $(\mathbb{R}, -dt^2)$ is represented by γ.*

Eschenburg uses the assumption of timelike completeness in a number of crucial ways. Consequently, the proof of the above theorem requires some new observations and techniques. At the same time, in devising a method of proof, we were strongly influenced by Eschenburg's work. In particular, our proof

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