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MANIFOLDS OF ALMOST NONNEGATIVE RICCI CURVATURE

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To investigate relations between Ricci curvature and topology of a Riemannian manifold is one of the main subjects in differential geometry. In this paper, we study the global structure of manifolds of almost nonnegative Ricci curvature by means of the convergence and collapsing phenomena of Riemannian manifolds.

Let M be a compact connected C^∞ Riemannian manifold of dimension n. A classical theorem of Bochner states that if the Ricci curvature $\operatorname{Ric}(M)$ of M is nonnegative, then the first Betti number $b_1(M)$ of M satisfies $b_1(M) \leq n$, where the equality takes place if and only if M is isometric to a flat torus. In [14], Gromov extended this result as follows: There is an $\varepsilon > 0$ depending on n and a given constant $D > 0$ such that if the diameter $d(M)$ and the Ricci curvature of M satisfy $d(M) \leq D$ and $\operatorname{Ric}(M) > -\varepsilon$, then the first Betti number of M is still bounded by n. Gallot [10] also gave an analytic proof.

Under an additional condition, an upper bound of sectional curvature, we determine the topological and global geometric structure. Let $\mathcal{M}(n, D)$ denote the family of compact Riemannian n-manifolds M with sectional curvature $K(M) \leq 1$ and diameter $d(M) \leq D$. Our first result is a topological classification by first Betti numbers.

Theorem 1. *There is an $\varepsilon > 0$ depending on n and D such that if $M \in \mathcal{M}(n, D)$ satisfy $\operatorname{Ric}(M) > -\varepsilon$, then M is a fiber bundle over a $b_1(M)$-torus. In particular, if $b_1(M) = n-1$, then M is diffeomorphic to an infranilmanifold, and if $b_1(M) = n$, then M is diffeomorphic to an n-torus.*

A manifold M is called an *infranilmanifold* if a finite covering space of M is a quotient of a simply connected nilpotent Lie group by its lattice. The special case $b_1(M) = n$ of Theorem 1 gives a partial affirmative answer to a conjecture of Gromov [14] (see the end of §2).

Our next goal is to describe the global geometric structure. To do this, we shall collapse Riemannian manifolds in the situation that the greatest lower bound of Ricci curvature tends to zero. In general cases, limit spaces

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