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A BOUNDARY OF THE SET OF THE RIEMANNIAN MANIFOLDS WITH BOUNDED CURVATURES AND DIAMETERS

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Dedicated to Professor Itiro Tamura on his sixtieth birthday

0. Introduction

In [12], Gromov introduced a metric (Hausdorff distance) on the class of all metric spaces. There, he proved the precompactness of the set consisting of the isometry classes of Riemannian manifolds with bounded curvatures and diameters. In this paper we shall study the structure of the closure of this set.

Definition 0.1. For a natural number n and $D \in (0, \infty]$, we let $\mathscr{M}(n, D)$ denote the set consisting of all isometry classes of compact Riemannian manifolds M such that

(0.2.1) the dimension of M is equal to n,

(0.2.2) the diameter of M is smaller than D,

(0.2.3) the sectional curvature of M is smaller than 1 and greater than -1.

The following problem is fundamental in the study of the Hausdorff distance on $\mathscr{M}(n, D)$.

Problem 0.3. (A) Determine the closure of $\mathscr{M}(n, D)$ with respect to the Hausdorff distance. (Hereafter $\mathscr{CM}(n, D)$ denotes the closure.)

(B) Let X_i $(i = 1, 2, \cdots)$ be a sequence of elements of $\mathscr{CM}(n, D)$. Suppose X_i converges to a metric space X with respect to the Hausdorff distance. Then, describe the relation between the topological structures of X_i and X.

Our main result on Problem 0.3(A) is Theorem 0.5 and those on Problem 0.3(B) are Theorems 0.12 and 10.1.

First we deal with Problem 0.3(A). Let $\mathscr{PM}_n$ denote the set of all pointed compact Riemannian manifolds (M, p) satisfying (0.2.1) and (0.2.3), and $\mathscr{CPM}_n$ the closure of $\mathscr{PM}_n$ with respect to the pointed Hausdorff distance (see 1.6). If $M \in \mathscr{CM}(n, D)$ then $(M, p) \in \mathscr{CPM}_n$ for each $p \in M$. We let $M(n, D, \mu)$ denote the set of the elements of $\mathscr{M}(n, D)$ whose injectivity radii

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