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SINGULARITIES OF A SIMPLE ELLIPTIC OPERATOR

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1. Introduction

Consider the operator A: $f \to -\Delta f + \Lambda(f)$ acting upon real functions of $x \in D$ with $f(x) = 0$ for $x \in \partial D$. $D \subset R^d$ is a d-dimensional domain. Λ: $R \to R$ is a convex function in the strict sense: $\Lambda''(f) > 0$. Let $0 < \lambda_1(D) < \lambda_2(D) \leqslant \lambda_3(D) \leqslant$ etc. $\uparrow \infty$ be the spectrum of $-\Delta \mid D$. Ambrosetti-Prodi [1] and Berger-Church [2] proved that if

$$-\infty < \Lambda'(-\infty) < \lambda_1(D) < \Lambda'(+\infty) < \lambda_2(D),$$

then A is a *global fold*, i.e., up to diffeomorphisms front and back, it has the form[1]

$$(f_1, f_2, f_3, \cdots) \to (f_1^2, f_2, f_3, \cdots).$$

This attractive result prompted McKean-Scovel [4] to study the simplest example in which $\Lambda'(R)$ crosses the whole spectrum of $-\Delta \mid D$: to wit, $d = 1$ and $D = (0,1)$ with $\Lambda(f) = f^2/2$, in which case A is the simple operator

$$B\colon f \to -f'' + f^2/2 \quad \text{subject to} \quad f(0) = f(1) = 0.$$

The present paper completes the description of the singularities of B.

Singular locus. This is the locus where the differential $dB = -d^2/dx^2 + f \equiv F$ is not $1:1$; it consists of a countable number of *sheets* M_n $(n \geqslant 1)$ indicated in Figure 1, marked off in function space by the vanishing of one of the (necessarily simple) eigenvalues

$$-\infty < \lambda_1(f) < \lambda_2(f) < \text{etc.} \to \infty$$

of F; in other words, $f \in M_n$ if and only if $Fe = 0$ has a solution $e = e_n$ with $e(0) = e(1) = 0$ and $n - 1$ more interior roots $e(r) = 0$ $(0 < r < 1)$. M_n is a smooth connected surface of codimension 1 with normal vector $\nabla \lambda_n = e_n^2$, the

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[1] $f \to (f_1, f_2, f_3, \cdots)$ is an assignment of coordinates in function-space.