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POISSON GEOMETRY OF THE PRINCIPAL SERIES AND NONLINEARIZABLE STRUCTURES

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Abstract

If $\mathfrak{g}$ is a Lie algebra (over $\mathbf{R}$), then the dual space $\mathfrak{g}^*$ carries a linear Poisson structure π_0 called the *Lie-Poisson structure*. The *linearization question* is whether a Poisson structure π defined near 0 on $\mathfrak{g}^*$ and having the same 1-jet as π_0 at 0 is equivalent to π_0 on a neighborhood of 0. The answer to the linearization question is *yes* if $\mathfrak{g}$ is semisimple and π is analytic or if $\mathfrak{g}$ is semisimple of compact type and π is C^∞ (Conn), but it can be *no* if $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R})$ and π is C^∞ (Weinstein). In this paper, we show that the answer can be *no* for a C^∞ π if $\mathfrak{g}$ is any semisimple algebra of noncompact type with real rank at least two. The cases of real rank one other than $\mathfrak{sl}(2, \mathbb{R})$ are still open.

The construction of nonlinearizable examples involves an analysis of the Poisson geometry of the subset of $\mathfrak{g}^*$ corresponding to the principal series representations. This analysis, in turn, relies on showing the functorial nature of the phase space for a classical particle in a Yang-Mills field (Sternberg, Weinstein). A by-product of our results is an analog in Poisson geometry to the correspondence between certain representations of $\mathfrak{g}$ and of the associated Cartan motion algebra (Gell-Mann, Hermann, Mukunda, Mackey, Dooley, and Rice).

0. Introduction

A *Poisson structure* on a manifold M may be defined as an antisymmetric contravariant tensor field (bivector) π for which the Poisson bracket operation on $C^\infty(M)$ defined by $\{f, g\} = [df \wedge dg, \pi]$ satisfies the Jacobi identity. The local classification problem for Poisson structures was reduced by the splitting theorem in [23] to the case where $\pi = 0$ at the point m_0 of interest. Near such a point, one may truncate the Taylor series of π at first order to obtain a linear Poisson structure of the form $\pi'_{ij}(x) = \sum c_{ij}^k x_k$, where the x_i are local coordinates centered around m_0. Such a structure is called a *Lie-Poisson structure*,

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