J. DIFFERENTIAL GEOMETRY
21 (1985) 151–162

HARMONIC MAPS TO SPHERES

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0. Introduction

Let M^k be an open Riemannian manifold. Let $S^n \subset \mathbf{R}^{n+1}$ be the familiar unit sphere. Here k and n are unrestricted positive integers, and for the rest of this paper, $\Sigma = \Sigma^{n-2} \subset S^n$ will denote an arbitrarily chosen, totally geodesic subsphere of codimension two. Our principal objects of study will be harmonic maps of the form $F\colon M \to S^n$, which avoid Σ. We have discovered that such maps possess special properties.

For example (leaving definitions momentarily aside), if M is compact, $F(M) \cap \Sigma = \varnothing$, and F is null-homotopic as a map to $S^n \sim \Sigma$, then F is constant (Theorem 1). When F is energy-minimizing, and bounded away from Σ (M typically noncompact), we obtain regularity and Liouville theorems. Namely, F is everywhere smooth (Theorem 2), and if $M = \mathbf{R}^k$, $n > 1$, actually constant (Theorem 4). Note these last two results are false without the boundedness assumption; e.g., the radial projection $\mathbf{R}^{n+1} \to S^n$ minimizes energy whenever $n > 6$ (and possibly even when $n > 2$) [8].

Our paper concludes with an appendix, containing a theorem on nodal (zero) sets of eigenfunctions on a compact Riemannian manifold. We include it here because it leads to an alternate proof of Theorem 1, and thereby casts a different, more geometric light on our results.

Let us make some of our terminology more precise. For further details, and usage not covered here, we recommend that the reader consult [3] or [7].

Consider a smooth map $F\colon M^k \to N^n$ between Riemannian manifolds, which has square summable first derivatives; that is, $F \in L^2_{1,\mathrm{loc}}(M, N)$. Associated to F, there is a function on M known as the *energy density*, and denoted here by $|DF|^2$. It is defined, at any point $x \in M$ by the formula

$$|DF|^2 = \sum_{i=1}^{k} \langle DF(e_i), DF(e_i) \rangle_N,$$

Received July 13, 1984.