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UNIQUENESS OF L^1 SOLUTIONS FOR THE LAPLACE EQUATION AND THE HEAT EQUATION ON RIEMANNIAN MANIFOLDS

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In this paper, our goal is to derive an optimal geometrical assumption on a complete Riemannian manifold which will ensure uniqueness properties for L^1 solutions of the Laplace equation and the heat equation. We say that a manifold has Ricci curvature with negative quadratic lower bound if there exists a point $x_0 \in M$ and a constant $C > 0$, such that the Ricci curvature at any point $x \in M$ satisfies

$$\text{Ric}(x) \geqslant -C(1 + r^2(x)), \tag{1}$$

where $r(x)$ denotes distance from x_0 to x. It turns out that the above Ricci curvature condition is the optimal assumption to guarantee uniqueness in both equations. In fact, we prove the following theorems.

Theorem 1. *Let M be a complete noncompact Riemannian manifold without boundary. If the Ricci curvature of M has a negative quadratic lower bound* (1), *then any L^1 nonnegative subharmonic function on M must be identically constant. In particular, any L^1 harmonic function on M must be identically constant.*

Theorem 2. *Let M be a complete noncompact Riemannian manifold without boundary. If the Ricci curvature of M has a negative quadratic lower bound* (1), *and if $v(x, t)$ is a nonnegative function defined on $M \times [0, \infty)$ satisfying*

$$\left(\Delta - \frac{\partial}{\partial t}\right) v(x,t) \geqslant 0, \qquad \int_M v(x,t)\, dx < \infty$$

for all $t > 0$, and

$$\lim_{t \to 0} \int_M v(x,t)\, dx = 0,$$

then $v(x, t) = 0$ for all $x \in M$ and $t \in (0, \infty)$.

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