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TOPOLOGICAL, AFFINE AND ISOMETRIC ACTIONS ON FLAT RIEMANNIAN MANIFOLDS

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Dedicated to the memory of Noel J. Hicks

Let M be a closed Riemannian flat manifold. It is well-known that Out $\pi_1(M)$, the outer automorphism group of the fundamental group of M, is isomorphic to the group $\pi_0(\mathcal{E}(M))$ of homotopy classes of self homotopy equivalences of M.

A homomorphism φ: $G \to \text{Out}\,\pi_1(M) \cong \pi_0(\mathcal{E}(M))$ is called an *abstract kernel* and is denoted by $(G, \pi_1(M), \varphi)$. A *geometric realization* of $(G, \pi_1(M), \varphi)$ *by a group of homeomorphisms* is a homomorphism $\hat{\varphi}$: $G \to \mathcal{H}(M)$, where $\mathcal{H}(M)$ is the group of homeomorphisms of M, so that $\hat{\varphi}$ composed with the natural homomorphism $\mathcal{H}(M) \to \text{Out}\,\pi_1(M)$ agrees with φ. This paper is concerned with the geometric realization problem when G is *finite* and M is *flat*, and is related to some of the ideas promulgated in [7].

In order that an abstract kernel has a geometric realization the kernel must have an "algebraic realization," [2, 2.2]. The Corollary to Lemma 1 characterizes the type of group extension which must exist if one is to find a geometric realization by an *effective* group of homeomorphisms on a closed aspherical manifold. Then Theorem 3 asserts that this necessary condition is also sufficient for an effective geometric realization on Riemannian flat manifolds. Because of flatness this realization can always be chosen to be a group of *affine* diffeomorphisms which, as we show in Theorems 3 and 6, is affinely equivalent to an isometric action on an affinely equivalent flat manifold. Thus it will follow that the finite groups which act effectively on M are isomorphic to those groups which act *isometrically* on manifolds *affinely equivalent* to M.

If, on the other hand, one is willing to sacrifice effectiveness one needs, as shown in the Corollary to Theorem 4, only the existence of *some* group

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