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SEPARABLE COORDINATES FOR THREE-DIMENSIONAL COMPLEX RIEMANNIAN SPACES

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1. Introduction

In this paper we study the problem of separation of variables for the equations

$$(1.1)\quad \begin{aligned} &\text{(a)}\quad \Delta_3\psi = \sum_{i,j=1}^{3} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial \psi}{\partial x^j}\right) = E\psi, \\ &\text{(b)}\quad \sum_{i,j=1}^{3} g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j} = E. \end{aligned}$$

Here $ds^2 = g_{ij}\, dx^i\, dx^j$ is a complex Riemannian metric, $g = \det(g_{ij})$, $g^{ij}g_{jk} = \delta^i_k$, $g_{ij} = g_{ji}$, and E is a nonzero complex constant. Thus (1.1)(a) is the eigenvalue equation for the Laplace-Beltrami operator on a three-dimensional complex Riemannian space whereas (1.1)(b) is the associated Hamilton-Jacobi equation.

We shall classify all metrics for which equations (1.1) admit solutions via separation of variables. Furthermore we shall indicate explicitly the group theoretic significance of each type of variable separation. The separation of variables problem for (1.1) has been studied by other authors, most notably by Stäckel [12], Robertson [11] and Eisenhart [14]. These authors were primarily concerned with systems for which the metric is orthogonal and in Stäckel form. Here, however, we classify all separable systems, orthogonal or not, in Stäckel form or not. Special emphasis is given to the nonorthogonal systems.

It is quite easy to show that (1.1)(b) admits (additive) separation of variables in every coordinate system for which (1.1)(a) admits a product separation and that in general (1.1)(b) separates in more systems than does (1.1)(a). However, we shall prove explicitly that in the cases where g_{ij} corresponds to flat space there is a one-to-one correspondence between separable systems for the equations. In these cases one can pass to Cartesian

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