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RIEMANNIAN SUBMERSIONS COMMUTING WITH THE LAPLACIAN

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1. Introduction

Let M and N be smooth Riemannian manifolds. Let $\Delta_M^p = d\delta + \delta d\colon \bigwedge^p(M) \to \bigwedge^p(M)$ denote the Laplace-Beltrami operator on the differential p-forms of M. Define the set

$$\Omega^p(M, N) = \{\varphi\colon M \to N \mid \varphi \text{ is a smooth surjective mapping with rank } \varphi_* \geq 1 \text{ and } \varphi^*\Delta_N^p A = \Delta_M^p \varphi^* A \text{ for all } A \in \bigwedge^p(N)\}$$

of pth Laplacian-commuting mappings. If $\Omega^p(M, N)$ is empty, it is said to be trivial. The condition on the rank is not necessary in defining $\Omega^0(M, N)$ because any surjective mapping $\varphi\colon M \to N$ with $\varphi^*\Delta_N f = \Delta_M \varphi^* f$ for all smooth functions f on N satisfies rank $\varphi_* = n = \dim N$. In this paper, we ask for the mappings contained in $\Omega^p(M, N)$. Watson [4] showed that $\varphi\colon M \to N$ is contained in $\Omega^0(M, N)$ if and only if it is a harmonic Riemannian submersion. He also proved that the nontriviality of $\Omega^p(M, N)$, $p \geq 0$, implies that the elements of $\Omega^p(M, N)$ are Riemannian submersions. We therefore ask for the Riemannian submersions which commute with the Laplacian. It is an immediate consequence of our main result that $\Omega^1(M, N) = \Omega^2(M, N) = \cdots = \Omega^n(M, N)$.

In § 2, the basic facts of a Riemannian submersion will be described, especially its structure tensor. Several relations between the curvature tensors of M and N and the structure tensor are given in § 3 .The set $\Omega^1(M, N)$ is studied in § 4, and in the last section the set $\Omega^p(M, N)$, $p \geq 2$, is examined.

2. Riemannian submersions

Let M (resp. N) be an m (resp. n)-dimensional manifold with Riemannian metric ds_M^2 (resp. ds_N^2), and let $\varphi\colon M \to N$ be a Riemannian submersion. Then we may assume $n < m$; for, if $m = n$, a Riemannian submersion (Riemannian covering) commutes with the Laplacian [4]. We choose local forms $\omega_1, \cdots, \omega_m$ on M and $\theta_1, \cdots, \theta_n$ on N such that $ds_M^2 = \Sigma\omega_a^2$, $ds_N^2 = \Sigma\theta_i^2$, and

$$\varphi^*(\theta_i) = \omega_i\,, \qquad i = 1, \cdots, n\,. \tag{2.1}$$

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