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THE SPECTRUM OF THE LAPLACIAN ON A MANIFOLD OF NEGATIVE CURVATURE. I

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1. Introduction

Let M be a simply connected complete two-dimensional Riemannian manifold. On M we have the Laplacian, a self-adjoint negative semidefinite operator on the Hilbert space $L^2(M)$. In case the curvature is everywhere $\leq -k^2 < 0$, it has been shown [1] that $\lambda_1 \geq k^2/4$, where λ_1 is the lower bound of the spectrum of the negative Laplacian.

The purpose of this note is to determine more accurate bounds for λ_1. We assume the following conditions on M:

(A) M possesses a global system of geodesic polar coordinates with respect to some point $0 \in M$.

Thus we are considering R^2 with a Riemannian metric of the form $ds^2 = dr^2 + G(r, \theta)^2 d\theta^2$ where $G(0^+, \theta) = 0$, $G_r(0^+, \theta) = 1$.

(B) $G_r(r, \theta) \geq 0$; $G(r, \theta) \geq g(r)$ where $g(r)$ is nondecreasing with $\lim_{r\to\infty} g(r) = \infty$.

Both of these conditions are satisfied in case the curvature is everywhere nonpositive. Finally we need the technical condition

(C) $|(G_{rr}/G_r)_\theta| \leq$ const, when $r \geq r_0$, $0 \leq \theta \leq 2\pi$.

Our main result states that

$$\inf_M (G_r/G) \leq \sqrt{4\lambda_1} \leq \inf_{0\leq\theta\leq 2\pi} \varlimsup_{r\to\infty} (G_{rr}/G_r) . \tag{1.1}$$

This result shows, for instance, that when the curvature is constant and equal to $-k^2$, then $\lambda_1 = \frac{1}{4}k^2$; no explicit calculations with special functions are needed in our approach.

To prove the lower half of (1.1) we modify the methods used in [1]. To obtain the upper bound we first obtain a comparison function and apply the variational characterization of λ_1. It is shown that if G_{rr}/G_r satisfies an upper bound on a sufficiently thick sector, then a corresponding upper bound can be obtained.

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