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SOME PROPERTIES OF $\underline{k}$-FLAT MANIFOLDS

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Introduction

In a recent paper [18], P. Molino, studying the characteristic classes of flat G-structures, remarked that his results more generally apply to the G-structures, which are locally equivalent to those defined, on a Lie group K, by action, under a group of automorphisms of the Lie algebra $\underline{k}$, on the left invariant parallelism of K. He named them "G-structures lisses".

Here we present some properties of this sort of structures, which we call $\underline{k}$-flat structures (structures lisses de type $\underline{k}$). Their behaviour is indeed led by a sort of flatness, where the usual abelian (i.e., torsion free) model is replaced by another one which is characterized by a canonical torsion, and is given by the local geometry of the Lie group K. Thus a $\underline{k}$-flat manifold admits $\underline{k}$, in a reasonable sense, as its Lie algebra (see [4]).

By defining morphisms so as to respect these Lie algebras, we obtain a category of *-flat manifolds which admits the category of Lie groups as a subcategory. Furthermore, the Lie groups are precisely the "tangent objets" in the category of *-flat manifolds. We also notice that the subcategory consisting in $\boldsymbol{R}^n$-flat manifolds is but the usual C^∞ category (§ 3).

Another aspect of the theory is the G-structural one. In this direction, we use a special kind of connections which generalizes the Cartan-Schouten connections on Lie groups [10].

Our main results are the characterization of $\underline{k}$-flat manifolds whose structural group is of finite type (if $\underline{k}$ is a semi-simple Lie algebra, this is always the case): they are discrete quotients of some open set of the Lie group K (§ 4), and the fact that if $\underline{k}$ is a reductive or a nilpotent Lie algebra, then any formal $\underline{k}$-flat structure on a manifold is $\underline{k}$-flat (§ 8).

§ 2 deals with a weak notion of $\underline{k}$-flatness which seems to us not to be lacking of interest. In §§ 5 and 6, we look for the polynomial vector fields, so useful in the study of flatness, and see them in a special kind of sub-structure (strict $\underline{k}$-flatness). Some special cohomological properties of $\underline{k}$-flatness are pointed out in § 7.

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Notation. Let $\underline{k}$ denote a real n-dimensional Lie *algebra*. A basis (e_1,

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