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SUBSCALAR PAIRS OF METRICS AND HYPERSURFACES WITH A NONDEGENERATE SECOND FUNDAMENTAL FORM

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0. Introduction

In this paper we establish an integral formula which holds on any compact oriented manifold without boundary equipped with a pair of Riemannian metrics. The natural assumptions needed to exploit this formula depend on positivity properties of a quadratic form constructed from the difference tensor of the Levi-Civita connections. We call two metrics satisfying this positivity property a subscalar pair.

The results are first applied to prove that subscalar pairs of Einstein metrics inducing the same element of volume are isometric. This generalizes a result of Munzner [10] on volume-preserving maps of the two-sphere in euclidean space.

Next we study the pseudo-Riemannian geometry of a hypersurface with a nondegenerate second fundamental form. In particular, we give a geometric interpretation of the rank of the difference tensor of the first and second fundamental forms, and establish local rigidity theorems on hypersurfaces with a given second fundamental form. In order to establish global results we assume that the hypersurfaces are convex, which for us means the second fundamental form of each convex hypersurface is negative definite. Under this assumption we give characterizations of the euclidean sphere in terms of various integral inequalities and prove a uniqueness theorem characterizing spheres as the only compact convex solutions of a differential inequality of 4th order in the derivatives of the position vector.

Finally we study the third fundamental form geometry of a convex hypersurface and prove that two compact convex hypersurfaces having the same second fundamental form and Gauss-Kronecker curvature differ by a rigid motion. This generalizes a result of Grove [6] on convex surfaces.

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