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METRICS AND ISOMETRIC EMBEDDINGS OF THE 2-SPHERE

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Since any compact C^2 two-dimensional submanifold of euclidean 3-space R^3 must have positive Gaussian curvature at some point, it follows that the 2-torus with flat metric and the compact orientable 2-manifolds of genus greater than 1 with metrics of everywhere negative curvature have no C^2 isometric embeddings in R^3. Of course, compact non-orientable 2-manifolds cannot be embedded in R^3 for topological reasons. A manifold of dimension $d > 2$ always admits a metric for which there is no isometric embedding in R^{d+1}: There certainly exists a metric such that all sectional curvatures are negative at some point of the manifold, by a standard extension argument for Riemannian metrics defined in a neighborhood of a point. There exists no C^2 isometric embedding in R^{d+1} for such a metric and in fact no C^2 isometric embedding in R^{d+1} of any neighborhood of a point of a d-dimensional manifold, $d > 2$, where all sectional curvatures are negative, as the expression for the sectional curvature of hypersurfaces in terms of the eigenvalues of the second fundamental form shows immediately.

The reasoning used in the cases already discussed fails to apply to any metric on the 2-sphere S^2, since $d = 2$ and the Gauss-Bonnet theorem guarantees at least one point of positive curvature for any given C^2 metric on the sphere. The purpose of this article is to exhibit a C^∞ metric on S^2 for which there is no C^2 isometric embedding in R^3. The proof of the non-existence of a C^2 embedding of S^2 in R^3 isometric for this metric is based on the analysis of the structure of flat submanifolds of R^3 given in Hartman and Nirenberg [1] (see also Massey [2]).

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1. The following results on flat submanifolds of R^3 will be used to show that the metric on S^2 constructed in § 2 has no C^2 isometric embedding in R^3.

Lemma 1 (*Hartman-Nirenberg*). *Let X be a C^2 surface with zero Gaussian curvature in R^3 with simple, nonsingular projection $P|X\colon X \to D_1$ onto a connected open set D_1 in the xy-plane, where $P : R^3 \to R^2$ is the canonical orthogonal*

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