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AN ANALYTIC PROOF OF RIEMANN-ROCH-HIRZEBRUCH THEOREM FOR KAEHLER MANIFOLDS

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1. Introduction

Let X be a compact complex manifold of (complex) dimension n, and ξ a holomorphic vector bundle over X. We shall denote by $\Omega(\xi)$ the sheaf of germs of holomorphic sections of ξ, and by $H^i(X, \Omega(\xi))$ the i-th cohomology group of the space X with coefficients in the sheaf $\Omega(\xi)$. Then $H^i(X, \Omega(\xi))$ are finite dimensional vector spaces over the field C of complex numbers, and $H^i(X, \Omega(\xi)) = 0$ for $i > n$. Let $\dim H^i(X, \Omega(\xi))$ denote the dimension of the vector space $H^i(X, \Omega(\xi))$, and $\chi(X, \Omega(\xi))$ be the Euler-Poincaré characteristic defined by the formula

$$\chi(X, \Omega(\xi)) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, \Omega(\xi)) .$$

Let $\mathscr{T}(X)$ be the Todd class of the complex tangent boundle $T(X)$ of X, and $\operatorname{ch}(\xi)$ the Chern character of the holomorphic vector bundle ξ. Then the Riemann-Roch-Hirzebruch theorem can be stated as follows.

Theorem 1.1. *The Euler-Poincaré characteristic* $\chi(X, \Omega(\xi))$ *can be expressed in terms of* $\operatorname{ch}(\xi)$ *and* $\mathscr{T}(X)$:

$$\chi(X, \Omega(\xi)) = [\operatorname{ch}(\xi)\mathscr{T}(X)]_{2n}[X] . \tag{1.1}$$

Formula (1.1) can be interpreted as follows: $\operatorname{ch}(\xi)$ and $\mathscr{T}(X)$ are elements of $H^*(X, Z) \otimes \mathbf{Q}$. If the multiplication is considered as the cup product, then $\operatorname{ch}(\xi)\mathscr{T}(X)$ defines an element of $H^*(X, Z) \otimes \mathbf{Q}$, and hence its $2n$-th component defines an element of $H^{2n}(X, Z) \otimes \mathbf{Q}$. The value of this element on the $2n$-dimensional cycle of X determined by the natural orientation is equal to $\chi(X, \Omega(\xi))$.

In this paper we shall give an analytic proof of this theorem under the assumption that X is a Kaehler manifold. We start with the following observations. Let η denote the complex vector bundle $\wedge(T^*(X)) \otimes C$, $T^*(X)$ being the cotangent bundle of X. Then η has a canonical direct sum decomposition

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