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ON A THEOREM OF F. SCHUR

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Let (M, g) be a C^4 Riemann manifold, $G_2(M)$ the Grassmann bundle of 2-planes on M, and $K\colon G_2(M) \to R$ the sectional curvature function. Let $\pi\colon G_2(M) \to M$ denote the canonical projection. Recall the theorem of F. Schur: if dimension $M \geq 3$, and $K|_{\pi^{-1}(p)} = \phi(p)$ for some $\phi\colon M \to R$, then (M, g) is of constant curvature. We shall view this theorem in the following setting:

Definition 1. Two Riemann manifolds (M, g), $(\overline{M}, \bar{g})$ are called *homocurved* if there exist a $1-1$ onto diffeomorphism $F\colon M \to \overline{M}$ and a function $\phi\colon M \to R$ such that for every $p \in M$ and $\sigma \in \pi^{-1}(p)$ we have

$$K(\sigma) = \phi(p)\overline{K}(F_*\sigma) ,$$

where $\overline{K}$ denotes the sectional curvature function of $(\overline{M}, \bar{g})$.

Definition 2. Homocurved manifolds are called homothetic (resp. *strongly homothetic*) if the corresponding $\phi \equiv$ constant (resp. F is a homothety).

'Strongly homothetic' clearly implies 'homothetic'. Converse is not true in general, e.g., consider the nonhomothetic conformal maps of constant curvature spaces. Schur's theorem says that a Riemann manifold of dimension ≥ 3 which is homocurved to a manifold of constant curvature is homothetic to it. A well-known fact about Einstein spaces is that a manifold homocurved to an Einstein manifold is homothetic to it.

Now we ask: *does "homocurved" imply "homothetic" in general*? We shall show that *generically* the answer to this question is yes.

Henceforth our standard situation will be the one described in Definition 1. *Throughout we shall use the notation and conventions of* [2].

Proposition 1. *Suppose that (M, g) is of dimension ≥ 3 and nowhere of constant curvature, i.e., on no nonempty open subset of M, $K \equiv$ constant. Then (M, g), $(\overline{M}, \bar{g})$ are conformal.*

Proof. This follows immediately from the general theorem of [2, § 2].

Proposition 2. *Suppose that (M, g) is of dimension ≥ 4 and nowhere conformally flat (cf.* [2, § 3]). *Then $\overline{R} = F_* R$, where $\overline{R}$ denotes the curvature tensor of $(\overline{M}, \bar{g})$.*

Proof. Identify M with $\overline{M}$ via F and consider the corresponding conformal deformation of the metric: $g \to F^*\bar{g} =$ (which we again denote by) $\bar{g} = f \cdot g$ where $f\colon M \to R$ is a positive real-value function. "Homocurved" implies

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