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THE EXISTENCE OF SPECIAL ORTHONORMAL FRAMES

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On an n-dimensional Riemann manifold M, the Laplace operator Δ on functions can be written (locally) in the form

$$\Delta = \sum_{i=1}^{n} X_i \circ X_i ,$$

where $\{X_1, \cdots, X_n\}$ is a (local) frame, if and only if the frame is orthonormal and $\operatorname{div} X_i = 0$, $i = 1, \cdots, n$. In Theorem 2.1, we formulate a condition relating the existence of special orthonormal frames to the Riemann curvature tensor. In Theorem 3.6, we show that the stronger condition: X_i is a Killing vector field, $i = 1, \cdots, n$, which implies $\operatorname{div} X_i = 0$, requires that M be Riemannian locally symmetric. It is further shown that most simply connected irreducible Riemannian symmetric spaces cannot have orthonormal frames consisting of Killing vector fields and that the spheres S^n have such frames if and only if $n = 1, 3$, or 7.

1. Introduction and motivation

Let M be an n-dimensional Riemannian manifold of class C^∞ with metric tensor g. The Riemannian connection ∇ is characterized by the conditions

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{1}$$

for all vector fields X, Y, and Z

(the connection is a metric connection) and

$$[X, Y] = \nabla_X Y - \nabla_Y X \tag{2}$$

for all vector fields X and Y

(the connection is torsionless).

Let $\{X_1, \cdots, X_n\}$ be an orthonormal frame on an open set U of M, that is,

$$g(X_i, X_j) = \delta_{ij} , \qquad i, j = 1, \cdots, n .$$

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