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HOMOGENEOUS SPACES DEFINED BY LIE GROUP AUTOMORPHISMS. II

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7. Noncompact coset spaces defined by automorphisms of order 3

We will drop the compactness hypothesis on G in the results of §6, doing this in such a way that problems can be reduced to the compact case. This involves the notions of reductive Lie groups and algebras and Cartan involutions.

Let $\mathfrak{G}$ be a Lie algebra. A subalgebra $\mathfrak{K} \subset \mathfrak{G}$ is called a *reductive subalgebra* if the representation $ad_{\mathfrak{G}}|_{\mathfrak{K}}$ of $\mathfrak{K}$ on $\mathfrak{G}$ is fully reducible. $\mathfrak{G}$ is called *reductive* if it is a reductive subalgebra of itself, i.e. if its adjoint representation is fully reducible. It is standard ([11, Theorem 12.1.2, p. 371]) that the following conditions are equivalent:

(7.1a) $\mathfrak{G}$ is reductive,

(7.1b) $\mathfrak{G}$ has a faithful fully reducible linear representation, and

(7.1c) $\mathfrak{G} = \mathfrak{G}' \oplus \mathfrak{Z}$, where the derived algebra $\mathfrak{G}' = [\mathfrak{G}, \mathfrak{G}]$ is a semisimple ideal (called the "semisimple part") and the center $\mathfrak{Z}$ of $\mathfrak{G}$ is an abelian ideal.

Let $\mathfrak{G} = \mathfrak{G}' \oplus \mathfrak{Z}$ be a reductive Lie algebra. An automorphism σ of $\mathfrak{G}$ is called a *Cartan involution* if it has the properties (i) $\sigma^2 = 1$ and (ii) the fixed point set $\mathfrak{G}'^{\sigma}$ of $\sigma|_{\mathfrak{G}'}$ is a maximal compactly embedded subalgebra of $\mathfrak{G}'$. The whole point is the fact ([11, Theorem 12.1.4, p. 372]) that

(7.2) *Let $\mathfrak{K}$ be a subalgebra of a reductive Lie algebra $\mathfrak{G}$. Then $\mathfrak{K}$ is reductive in $\mathfrak{G}$ if and only if there is a Cartan involution σ of $\mathfrak{G}$ such that $\sigma(\mathfrak{K}) = \mathfrak{K}$.*

Let G be a Lie group. We say that G is *reductive* if its Lie algebra $\mathfrak{G}$ is reductive. Let K be a Lie subgroup of G. We say that K is a *reductive subgroup* if its Lie algebra $\mathfrak{K}$ is a reductive subalgebra of $\mathfrak{G}$. Let σ be an automorphism of G. We say that σ is a *Cartan involution* of G if σ induces a Cartan involution of $\mathfrak{G}$.

Let G be a reductive Lie group, and K a closed reductive subgroup such that G acts effectively on $X = G/K$. Choose a Cartan involution σ of $\mathfrak{G}$ which preserves $\mathfrak{K}$, and consider the decomposition into (± 1)-einspaces of σ:

$$(7.3a) \qquad \mathfrak{G} = \mathfrak{G}^{\sigma} + \mathfrak{M}\,, \qquad \mathfrak{K} = \mathfrak{K}^{\sigma} + (\mathfrak{K} \cap \mathfrak{M})\,.$$

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