J. DIFFERENTIAL GEOMETRY
52 (1999) 117-144

CONVEX BODIES OF CONSTANT BRIGHTNESS AND A NEW CHARACTERISATION OF SPHERES

FRANÇOIS HAAB

Abstract

Nakajima showed that if a convex body in $\mathbb{R}^3$ satisfying certain smoothness conditions has constant width and constant brightness, then it is a ball. This work extends Nakajima's result to higher dimensions. We prove that if K is a convex body in $\mathbb{R}^d$ of class C^2_+ with constant i-brightness and constant j-brightness, then K is a ball. We also generalize this result to relative differential geometry.

1. Introduction

In this article we obtain a new characterisation of balls among convex bodies in $\mathbb{R}^n$ of class C^2_+. A convex body K in $\mathbb{R}^n$ is of *class* C^k_+, $k \geq 2$, if its boundary, denoted by $\operatorname{bd} K$, is a hypersurface of class C^k and if the Gauss-Kronecker curvature of $\operatorname{bd} K$ is positive at any point $x \in \operatorname{bd} K$.

A convex body K in the Euclidean space $\mathbb{R}^n$ is of *constant k-brightness* ([10, 3.3.10]) or of *constant outer k-measure* ([6, p.81]) if all its orthogonal projections on k-dimensional linear subspaces of $\mathbb{R}^n$ have the same k-volume. When $k = 1$, K is of *constant width* (Similary when $k = n-1$, K is of *constant brightness*).

If K is a convex body in $\mathbb{R}^n$ with constant width and constant k-brightness for a given $k > 1$, is K a ball ?

This classical question ([14, p.368]; [6, p.82]; [7, problem A10]) is called the *Nakajima problem* by Goodey, Schneider and Weil in [13]. In 1926, Nakajima [18] answered positively if $n = 3$ and $k = 2$ and K is of class C^2_+ (see also Bonnesen and Fenchel's book of 1934 ([3, p.140]) or ([10, 3.3.20]) for a more recent viewpoint).

Received May 15, 1998, and, in revised form, February 23, 1999.