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BEHAVIOR OF GEODESIC-LENGTH FUNCTIONS ON TEICHMÜLLER SPACE

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Abstract

Let $\mathcal{T}$ be the Teichmüller space of marked genus g, n punctured Riemann surfaces with its bordification $\overline{\mathcal{T}}$ the *augmented Teichmüller space* of marked Riemann surfaces with nodes, [**Abi77**, **Ber74**]. Provided with the WP metric, $\overline{\mathcal{T}}$ is a complete $CAT(0)$ metric space, [**DW03**, **Wol03**, **Yam04**]. An invariant of a marked hyperbolic structure is the length ℓ_α of the geodesic α in a free homotopy class. A basic feature of Teichmüller theory is the interplay of two-dimensional hyperbolic geometry, Weil-Petersson (WP) geometry and the behavior of geodesic-length functions. Our goal is to develop an understanding of the intrinsic local WP geometry through a study of the gradient and Hessian of geodesic-length functions. Considerations include expansions for the WP pairing of gradients, expansions for the Hessian and covariant derivative, comparability models for the WP metric, as well as the behavior of WP geodesics, including a description of the Alexandrov tangent cone at the augmentation. Approximations and applications for geodesics close to the augmentation are developed. An application for fixed points of group actions is described. Bounding configurations and functions on the hyperbolic plane is basic to our approach. Considerations include analyzing the orbit of a discrete group of isometries and bounding sums of the inverse square exponential-distance.

1. Introduction

1.1. Background. Collections of geodesic-length functions provide local coordinates for Teichmüller space. Gardiner provided a formula for the differential of geodesic-length and for the WP gradient of geodesic-length, [**Gar75**]. In [**Wol82**] a geometric and analytic description of the deformation vector field for the infinitesimal Fenchel-Nielsen right twist t_α revealed the twist-length duality formula $2t_\alpha = i \operatorname{grad} \ell_\alpha$, [**Wol82**, Thrm. 2.10]. The formula combines with the calculation of the twist

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