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ON THE REGULARITY OF SOLUTIONS TO A GENERALIZATION OF THE MINKOWSKI PROBLEM

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The Minkowski Problem concerns the existence, uniqueness, and regularity of closed convex hypersurfaces whose Gauss curvature (as a function of the outer normals) is preassigned. Major contributions to this problem were made by Minkowski [28], [29], Aleksandrov [2], [4], Lewy [23], [24], Nirenberg [30], Calabi [9], Pogorelov [34], [35], and Cheng and Yau [10]. Variants of the Minkowski Problem were presented by Gluck [16] and Singer [41]. The survey of Gluck [17] still serves as an excellent introduction to the problem. In this article we consider a generalization of the Minkowski Problem.

We first recall the analytic formulation of the classical Minkowski Problem. Suppose $u = (u^1, \cdots, u^{n-1})$ are smooth local coordinates on the standard unit sphere, S^{n-1}, in Euclidean n-space $\mathbb{R}^n$, and $e = e_{ij}\, du^i\, du^j$ is the first fundamental form of S^{n-1}. The Einstein convention on summation (over repeated lower and upper indices) is presumed everywhere (with Latin indices running from 1 to $n-1$). Let Γ^k_{ij} denote the Christoffel symbols of the second kind for the metric e. For $h \in C^2(S^{n-1})$, let

$$\nabla_{ij} h = \partial_{ij} h - \Gamma^k_{ij} \partial_k h,$$

where

$$\partial_k h = \frac{\partial h}{\partial u^k}, \qquad \partial_{ij} h = \frac{\partial^2 h}{\partial u^i\, \partial u^j},$$

and define the operator N by

$$N(h) = \frac{\det(\nabla_{ij} h + e_{ij} h)}{\det(e_{ij})}.$$

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